Homework Assignment #2

8.7 Show that $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}$ are idempotent.

Answer: By direct computation, we find

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}^2 = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}^2 = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}.$$

8.20 Invert the coefficient matrix to solve the following systems of equations:

$$a)\ \begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 + x_2 &= 3; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 + x_2 &= 4 \\ 6x_1 + 2x_2 + 6x_3 &= 20 \\ -4x_1 - 3x_2 + 9x_3 &= 3; \end{aligned} \qquad c)\ \begin{aligned} 2x_1 + 4x_2 &= 2 \\ 4x_1 + 6x_2 + 3x_3 &= 1 \\ -6x_1 - 10x_2 &= -6. \end{aligned}$$

Answer:

a) The inverse is $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, which yields solution $A^{-1}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b) The inverse is $A = \begin{pmatrix} -6 & 3/2 & -1 \\ 13 & -3 & 2 \\ 5/3 & -1/3 & 1/3 \end{pmatrix}$, yielding solution $A^{-1}\begin{pmatrix} 4 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

c) The inverse is $A = \begin{pmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 5/6 & 1/3 & 1/2 \end{pmatrix}$, yielding solution $A^{-1}\begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

8.27

a) Prove that $(AB)^k = A^k B^k$ if $AB = BA$.

b) Show that $(AB)^k \neq A^k B^k$ in general.

c) Conclude that $(A+B)^2$ does not equal $A^2 + 2AB + B^2$ unless $AB = BA$.

Answer:

a) Now $(AB)^k = (AB)(AB)\cdots(AB)$. We reassociate to obtain $A(BA)(BA)\cdots(BA)B = A(AB)\cdots(AB)B$. Continue to reassociate and move the A's to the left and B's to the right to obtain $A^k B^k$.

b) Here is an example. Let

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \text{ and } (AB)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}$$

but

$$A^2B^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \neq (AB)^2.$$

In fact, it is easy to see that $A^{2k}B^{2k} = 2^k I_2$ while $(AB)^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 4^k \end{pmatrix}$ and that $A^{2k+1}B^{2k+1} = 2^k AB$ while $(AB)^{2k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 4^k \end{pmatrix} AB \neq A^{2k+1}B^{2k+1}$.

In fact, if $(AB)^2 = A^2B^2$, we have $ABAB = AABB$. If both A and B are invertible, we pre-multiply by A^{-1} and post-multiply by B^{-1} to find $BA = AB$. This doesn't work if A or B is not invertible.

c) Suppose $(A+B)^2 = A^2+2AB+B^2$. This expands to $A^2+BA+AB+B^2 = A^2+2AB+B^2$. Canceling the common terms, we find $BA = AB$. The derivation is reversible, so $(A+B)^2 = A^2+2AB+B^2$ if and only if $AB = BA$.

9.12 Use Cramer's rule to compute x_1 and x_2 in Example 9.4.

Answer: The system in 9.4 is

$$\begin{pmatrix} 1 & 1 & 1 \\ 12 & 2 & -3 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}.$$

As in Example 9.4, the determinant of the coefficient matrix A is 35. We compute the determinants of

$$B_1 = \begin{pmatrix} 0 & 1 & 1 \\ 5 & 2 & -3 \\ -4 & 4 & 1 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 1 & 0 & 1 \\ 12 & 5 & -3 \\ 3 & -4 & 1 \end{pmatrix}$$

Then $|B_1| = 35$ and $|B_2| = 50$, so $x_1 = 35/35 = 1$ and $x_2 = -70/35 = -2$.

26.9 Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 6 & 4 \\ 1 & 1 \end{pmatrix}$.

a) Show that $\det(A+B) \neq \det A + \det B$.

Answer: Here

$$A + B = \begin{pmatrix} 6 & 4 \\ 2 & 2 \end{pmatrix}$$

implying $\det(A+B) = 12 - 8 = 4$. But $\det A = 2 - 1 = 1$ and $\det B = 4 - 3 = 1$, so $\det(A+B) \neq \det A + \det B$.

b) Show that $\det A + \det B = \det C$ and relate this to Fact 26.6.

Answer: Here $\det C = 6 - 4 = 2 = 1 + 1 = \det A + \det B$. We note that A, B, and C all have the same bottom row, and that the top row of C is the sum of the top rows of A and B. Hence $\det A + \det B = \det C$ by Fact 26.6.

26.14 Find the exact values of k which make each of the following matrices singular:

$$a) \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix} \qquad b) \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}.$$

Answer:

a) In this case, the determinant is $1 - k^2$, so the matrix will be singular if and only if $k^2 = 1$, which corresponds to $k = \pm 1$.

b) Here the determinant is $k^3 - 3k + 2$, which factors to $(k-1)(k^2+k-2) = (k-1)^2(k+2)$. The matrix is singular if and only if $k \in \{1, -2\}$.