Homework Assignment #3

10.20 Use vector notation to prove that the diagonals of a rhombus are orthogonal to each other. See Figure 10.23.

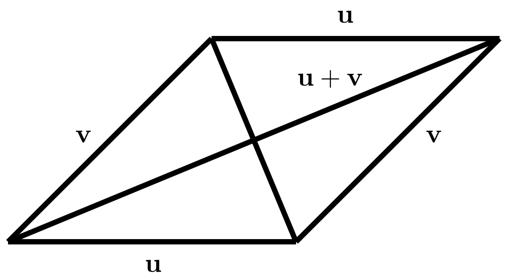


Figure 10.23: *If* $\|\mathbf{u}\| = \|\mathbf{v}\|$*, this quadrilateral is a rhombus.*

Answer: The diagonals are $(\mathbf{u}+\mathbf{v})$ and $(\mathbf{v}-\mathbf{u})$. We compute the dot product $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{v}-\mathbf{u}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} = \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2$. Since $\|\mathbf{u}\| = \|\mathbf{v}\|$, the dot product is zero. This shows that the diagonals are orthogonal.

10.27 Show that the midpoint of $\ell(\mathbf{x}, \mathbf{y})$ occurs where $t = \frac{1}{2}$. In other words, if $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$, show that $\|\mathbf{x} - \mathbf{z}\| = \|\mathbf{y} - \mathbf{z}\|$.

Answer: Here $\|\mathbf{x} - \mathbf{z}\| = \|\frac{1}{2}\mathbf{x} - \frac{1}{2}\mathbf{y}\|$ and $norm \mathbf{y} - \mathbf{z} = \|\frac{1}{2}\mathbf{y} - \frac{1}{2}\mathbf{x}\|$. Since the norm is absolutely homogeneous of degree 1, the two expressions are equal.

10.41 Use Gaussian elimination to find the equation of the line which is the intersection of the planes $x+y-z = 4$ and $x + 2y + z = 3$.

Answer: The corresponding augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 3 \end{pmatrix}.$$

This yields reduced row echelon form

$$\begin{pmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & -1 \end{pmatrix}.$$

Thus $x = 5 + 3z$ and $y = -1 - 2z$. We can write this in parametric form as $(x, y, z)^{\mathbf{T}} = (5, -1, 0)^{\mathbf{T}} + t(3, -2, 1)^{\mathbf{T}}$.

12.14 Prove that the strictly positive orthant $\mathbb{R}^m_{++} \equiv \{(x_1, \ldots, x_m) : x_i > 0 \text{ for } i = 1, \ldots, m\}$ is an open subset of $\mathbb{R}^m_{++}$ by finding a formula for ϵ in terms of the x_i's.

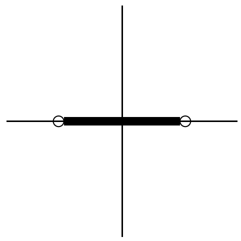
Answer: Let $x \in \mathbb{R}^m_{++}$. Let $\epsilon = \min\{x_1, \ldots, x_m\}$. Let $y \in B_\epsilon(x)$. Then $|y_i - x_i| \le \|y - x\| < \epsilon$. Thus $-\epsilon < y_i - x_i < \epsilon$, which implies $x_i - \epsilon < y_i$. But $x_i - \epsilon \ge 0$, so $y_i > 0$. Since this holds for every $i = 1, \ldots, m$, $y \in \mathbb{R}^m_{++}$.

12.21 For each of the following subsets of the plane, draw the set, state whether it is open, closed, or neither, and justify your answer in a word or two:

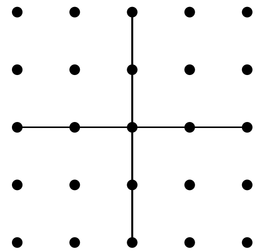
a) $\{(x, y) : -1 < x < +1, y = 0\}$, *b)* $\{(x, y) : x \text{ and } y \text{ are integers}\}$,

c) $\{(x, y) : x + y = 1\}$, *d)* $\{(x, y) : x + y < 1\}$, *e)* $\{(x, y) : x = 0 \text{ or } y = 0\}$.

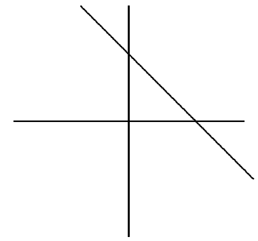
Answer:



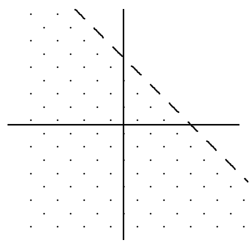
Set (a)



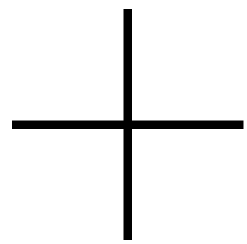
Set (b)



Set (c)



Set (d)



Set (e)

a) This set is neither open ($B_\epsilon(0,0)$ pokes out), nor closed (the limit points $(-1,0)$ and $(1,0)$ are not included).

b) This set is closed. Any convergent sequence with integral coordinates must eventually be constant since there is only one integer point within any distance $\epsilon < 1/2$

c) This set is not open ($B_\epsilon(1,0)$ pokes out). It is closed (if $f(x,y) = x+y$, f is continuous and the set is $f^{-1}(\{1\})$.).

d) This set is open (using the previous f, it is $f^{-1}(-\infty,0)$). It is not closed since $(1,-1/n)$ is in the set and converges to a point outside the set.

e) This set is not open ($B_\epsilon(0,0)$ pokes out), but is closed as the union of two closed sets (the coordinate axes).

12.32 Prove that the intersection of compact sets is compact and that the finite union of compact sets is compact. Show that the infinite union of compact sets need not be compact.

Answer: We will use the 'closed and bounded' definition of compact. We first consider the intersection of a collection of compact sets. We know that compact sets are closed, and any intersection of closed sets is closed. So any intersection of compact sets is also closed. Further, let r be a bound for one of the compact sets. Then r is also a bound for the intersection. The intersection is then both closed and bounded, hence compact.

Now consider the finite union of compact sets, C_k, $k = 1, \ldots, K$. Each set is closed, and we know the finite union of closed sets is closed, so $C = \cup_{k=1}^K C_k$ is closed. Let r_k bound C_k. Let $r = \max r_k$, which exists because we have a finite number of r_k's. Then r is a bound for C. Since C is closed and bounded, it is compact.

Finally, let $C_k = [-k,k] \subset \mathbb{R}$ and consider $C = \cup_{k=1}^\infty = \mathbb{R}$. This set is not bounded, and so not compact. The infinite union of compact sets need not be compact.