Homework Assignment #4

13.12 Write the following quadratic forms in matrix form:

a) $x_1^2 - 2x_1x_2 + x_2^2$,

b) $5x_1^2 - 10x_1x_2 - x_2^2$,

c) $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3$.

Answer: If we require the matrices to be symmetric, the solutions are:

$$a)\ x_1^2 - 2x_1x_2 + x_2^2 = \mathbf{x}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$$

$$b)\ 5x_1^2 - 10x_1x_2 - x_2^2 = \mathbf{x}^T \begin{bmatrix} 5 & -5 \\ -5 & -1 \end{bmatrix} \mathbf{x}$$

$$c)\ x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 - 6x_1x_3 + 8x_2x_3 = \mathbf{x}^T \begin{bmatrix} 1 & 2 & -3 \\ 2 & 2 & 4 \\ -3 & 4 & 3 \end{bmatrix} \mathbf{x}$$

13.21 Let $f\colon \mathbb{R}^k \to \mathbb{R}^1$ be continuous at the point $\mathbf{a} = (a_1, \dots, a_k)$. Consider the function $g\colon \mathbb{R}^1 \to \mathbb{R}^1$ defined by $g(t) = f(t, a_2, \dots, a_k)$. Show that g is continuous at a_1. This result implies that if f is continuous, its restriction to any line parallel to a coordinate axis is also continuous. However, the converse is not true. Consider the function $f(x, y) = xy^2/(x^2 + y^4)$. Show that $f_1(t) = f(t, a)$ and $f_2(t) = f(a, t)$ are continuous functions of t for each fixed a. Show that f itself is not continuous at $(0, 0)$. [Hint: Take a sequence on the diagonal.]

Answer: Let $t_n \to t$. Then $(t_n, a_2, \dots, a_k) \to (t, a_2, \dots, a_k)$. Since f is continuous, $g(t_n) = f(t_n, a_2, \dots, a_k) \to f(t, a_2, \dots, a_k) = g(t)$, showing continuity of g.

For the function $f(x, y) = xy^2/(x^2 + y^4)$ we define $f(0, 0) = 0$. Then $f_1(t) = ta^2/(t^2 + a^4)$. Clearly $f_1(t_n) \to f_1(t)$ whenever $t_n \to t$, even if $t = 0$. When $a = 0$, it reduces to 0 for $t \neq 0$. Since we have set $f(0, 0) = 0$, this is also continuous. The case of $f_2(t) = at^2/(a^2 + t^4)$ is similar.

The hint is not quite correct as one should use a parabola, not a line. Consider $f_3(t) = f(t^2, t) = 1/2$. This does not converge to 0 as $t \to 0$, so f is not continuous.

13.24 For each of the following functions, write h as the composition of two functions f and g: the image of f? Which ones are one-to-one? For those which are one-to-one, write the expression for the inverse. Which ones are onto?

a) $h(x) = \log(x^2 + 1)$;

b) $h(x) = (\sin x)^2$;

c) $h(x) = (\cos x^3, \sin x^3)$;

d) $h(x) = (x^2y)^3 + x^2y$;

Answer:

a) Let $f(x) = x^2 + 1$ and $g(y) = \log y$. Then $h = g \circ f$.

b) Let $f(x) = \sin x$ and $g(y) = y^2$. Then $h = g \circ f$.

c) Let $f(x) = x^3$ and $g(y) = (\cos y, \sin y)$. Then $h = g \circ f$.

d) Let $f(x, y) = x^2y$ and $g(z) = z^3 + z$. Then $h = g \circ f$.

14.4 Consider the production function $Q = 9L^{2/3}K^{1/3}$.

a) What is the output when $L = 1000$ and $K = 216$?

Answer: Here $1000 = 10^3$ and $216 = 6^3$, so $Q = 9(100)(6) = 5400$.

b) Use marginal analysis to estimate $Q(998, 216)$ and $Q(1000, 217.5)$.

Answer: Now $Q(998, 216) \approx 5400 - 2(\partial Q/\partial L)$ and $Q(1000, 217.5) \approx 5400 + 1.5(\partial Q/\partial K)$. Here $(\partial Q/\partial L)(1000, 216) = 3.6$ and $(\partial Q/\partial K)(1000, 216) = 25/3$, so the approximate values are $Q(998, 216) \approx 5392.8$ and $Q(1000, 217.5) \approx 5412.5$.

c) Use a calculator to compute these two values of Q to three decimal places and compare these values with your estimates in *b*.

Answer: According to the calculator, $Q(998, 216) \approx 5392.798$, a discrepancy of 0.002 and $Q(1000, 217.5) \approx 5412.471$, a discrepancy of 0.029.

d) How big must ΔL be in order for the difference between $Q(1000 + \Delta L, 216)$ and its linear approximation, $Q(1000, 216) + (\partial Q/\partial L)(1000, 216)\Delta L$, to differ by more than 2 units? (Plug increasing values of L into these two expressions.)

Answer: Here $Q(1000, 216) + (\partial Q/\partial L)(1000, 216)\Delta L = 5400 + 3.6\Delta L$ and $Q(1000 + \Delta L, 216) = 54(1000 + \Delta L)^{2/3}$. Using a spreadsheet, I found that at $\Delta L = 58.48$ the difference is just over 2 units (2.0004). The same was true at $\Delta L = -57$, where the difference was 2.0005.

14.17 Let $w(r, s)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^1$. Let $r = y - x$ and $s = y + x$. Let $F(x, y) = w\big(r(x, y), s(x, y)\big)$. Compute $\partial F/\partial x$ and $\partial F/\partial y$ in terms of $\partial w/\partial r$ and $\partial w/\partial s$.

Answer: We write (x, y) and (r, s) as column vectors. Then the Chain Rule yields:

$$\begin{aligned} dF = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} &= dw \times d\begin{pmatrix} r \\ s \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} \end{pmatrix} \times \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} & \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} \end{pmatrix}. \end{aligned}$$

14.27 Consider the production function $Q = K^{3/4}L^{3/4}$. Show that marginal productivity of each factor is diminishing. Show, however, that for any strictly positive input combination, if the input combination is doubled, then output more than doubles.

Answer: The marginal products are $\mathrm{MP}_K = \frac{\partial Q}{\partial K} = (3/4)K^{-1/4}L^{3/4}$ and $\mathrm{MP}_L = \frac{\partial Q}{\partial L} = (3/4)K^{3/4}L^{-1/4}$, which are decreasing in K and L, respectively.

However, if we double the inputs, so $Q(2K, 2L) = 2^{3/2}K^{3/4}L^{3/4} = 2^{3/2}Q(K, L) > 2Q(K, L)$, we more than double the output.