Homework Assignment #5

15.6 Consider the function $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3$.

a) If $x_1 = 6$ and $x_2 = 3$, find a y which satisfies $F(x_1, x_2, y) = 0$.

b) Does this equation define y as an implicit function of x_1 and x_2 near $x_1 = 6$, $x_2 = 3$?

c) If so, compute $\partial y/\partial x_1(6, 3)$ and $\partial y/\partial x_2(6, 3)$.

d) If x_1 increases to 6.2 and x_2 decreases to 2.9, estimate the corresponding change in y.

Answer:

a) The equation becomes $27 + y^3 = 0$, which is satisfied by $y = -3$ (this is the only real solution, but there are also two complex solutions, $y = 3e^{\pm 2i/3}$).

b) We calculate $\partial F/\partial y(1, 3, -3) = 3y^2(1, 3, -3) = 3 \cdot (-3)^2 = 27 \neq 0$. By the Implicit Function Theorem, y can be implicitly defined as a function of (x_1, x_2) in a neighborhood of $(1, 3)$.

c) Here $\frac{\partial y}{\partial x_1}(6, 3, -3) = -(\frac{\partial F}{\partial x_1}/\frac{\partial F}{\partial y})(6, 3, -3) = -(2x_1/3y^2)(1, 3, -3) = -4/9$ and $\frac{\partial y}{\partial x_2} = -(\frac{\partial F}{\partial x_2}/\frac{\partial F}{\partial y})(6, 3, -3) = (2x_2/3y^2)(6, 3, -3) = 6/27 = 2/9$.

d) The change is $\Delta y \approx -(4/9)\Delta x_1 + (2/9)\Delta x_2 = -(4/9)(.2) - (2/9)(.1) = -1/9 \approx -0.11$.

15.8 Consider the equation $x^3 + 3y^2 + 4xz^2 - 3z^2y = 1$. Does this equation define z as a function of x and y:

a) In a neighborhood of $x = 1$, $y = 1$?

b) In a neighborhood of $x = 1$, $y = 0$?

c) In a neighborhood of $x = 0.5$, $y = 0$? If so, compute $\partial z/\partial x$ and $\partial z/\partial y$ at this point.

Answer:

a) **No.** When $x = 1$ and $y = 1$, $4 + z^2 = 1$, which has no real solution z.

b) **No.** When $x = 1$ and $y = 0$, $1 + 4z^2 = 1$, implying $z = 0$. We calculate $\partial F/\partial z = 8xz - 6yz$. At $(x, y, z) = (1, 0, 0)$ $\partial F/\partial z = 0$. As this is not invertible, the Implicit Function Theorem does not apply.

In fact, if $x = 1 + \epsilon$ for small $\epsilon > 0$, and $y = 0$, the equation has no real solution. Hence there is no function.

c) **Yes.** Here $z^2 = 7/16$ so $z = \pm\sqrt{7}/4$. Then $\partial F/\partial z = 8xz - 6yz$ so $\partial F/\partial z|_{(1/2,0,7/8)} = 4z = \pm\sqrt{7}$. The partial derivatives are

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{10/4}{\pm\sqrt{7}} = \mp\frac{5}{2\sqrt{7}} = \mp\frac{5\sqrt{7}}{14} \approx \mp 0.945$$

and

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{-21/16}{\pm\sqrt{7}} = \pm\frac{21}{16\sqrt{7}} = \pm\frac{3\sqrt{7}}{16} \approx \pm 0.496$$

15.13 A firm uses x hours of unskilled labor and y hours of skilled labor each day to produce $Q(x, y) = 60x^{2/3}y^{1/3}$ units of output per day. It currently employs 64 hours of unskilled labor and 27 hours of skilled labor.

a) What is its current output?

b) In what direction (expressed as a unit vector) should it change (x, y) if it wants to increase output most rapidly?

c) The firm is planning to hire an additional hour and a half of skilled labor. Use calculus to estimate the corresponding change in unskilled labor that would keep its output at its current level.

Answer:

a) Its current output is $Q(64, 27) = 60(2^4)3 = 2880$.

b) Here $dQ^T = Q(2/3x, 1/3y)^T = 2880(1/96, 1/81)^T = (30, 35\frac{5}{9})^T$ is the direction of fastest increase. To express that as a unit vector, we divide it by $\|(30, 35\frac{5}{9})^T\| = 10\sqrt{1753}/9 \approx 46.52$, obtaining $1753^{-1/2}(27, 32)^T \approx (0.58, 0.69)^T$

c) Using the implicit function theorem, we find we may treat x as a function of y because $\partial Q/\partial x = 30 \neq 0$. Then $dx/dy = -(35\frac{5}{9})/30 = -32/27$. When we increase y by 1.5 we must decrease x by $(-32/27)(1.5) = -16/9$ in order to maintain output at its current level.

16.2 Let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form on $\mathbb{R}^n$. By evaluating Q on each of the coordinate axes in $\mathbb{R}^n$, prove that a necessary condition for a symmetric matrix to be positive definite (positive semidefinite) is that all the diagonal entries be positive (nonnegative). State and prove the corresponding result for negative and negative semidefinite matrices. Give an example to show that this necessary condition is not sufficient.

Answer: The corresponding result is the following theorem.

Theorem 1. *A necessary condition for for a symmetric matrix to be negative definite (negative semidefinite) is that all the diagonal entries are negative (nonpositive).*

Proof. Calculate $Q(\mathbf{e}^i) = a_{ii}$. This must be negative for all i if Q is negative definite, nonpositive for all i if Q is negative semidefinite.

To show the necessary conditions are not sufficient, consider

$$A_1 = \begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}.$$

Then A_1 satisifes the necessary conditions for negative definiteness, and A_2 satisfies the necessary conditions for negative semidefiniteness. But the corresponding quadratic forms take the values $Q_1(1,1) = 6$ and $Q_2(1,1) = 8$, showing that A_1 is not negative definite and that A_2 is not negative semidefinite.

29.9 Give an example to show that the interior of a connected set need not be connected.

Answer: Consider $S = \{(x, y) : |y| \leq |x|\}$. It is a connected set (in fact, is star-shaped relative to the origin). Its interior is $\operatorname{int} S = \{(x, y) : |y| < |x|\}$, which fails to be connected because the origin is not in $\operatorname{int} S$.

29.11 For each of the following subsets of $\mathbb{R}^2$, *a*) sketch the set and *b*) determine whether or not it is open, closed, compact, or connected. Give reasons for your negative answers to part *b*.

$$i)\ \{(x, y) : x = 0, y \geq 0\}, \qquad ii)\ \{(x, y) : 1 \leq x^2 + y^2 \leq 2\},$$
$$iii)\ \{(x, y) : 1 \leq x \leq 2\}, \qquad iv)\ \{(x, y) : x = 0 \text{ or } y = 0, \text{but not both}\}.$$

Answer: The graphs are below the answers.

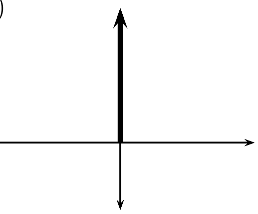
i) It is closed and connected. Any ball around $(0, 0)$ contains points outside the set, so it is not open. It is not compact because is it not bounded.

ii) It is closed, compact, and connected. It is not open because no open ball about $(1, 0)$ is contained in the set.

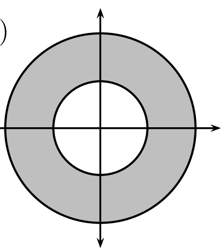
iii) It is closed and connected. It is not open because no ball around $(1, 0)$ is contained in the set. It is not compact because it is not bounded.

iv) It is *not* open, closed, compact, or connected. It is not open because it contains no open ball about the point $(1, 0)$. It is not closed because the limit point $(0, 0) = \lim(1/n, 0)$ is not in the set. It is not compact because it is not closed. Finally, it is not connected because the open sets $U = \{(x, y) : x + y > 0\}$ and $V = \{(x, y) : x + y < 0\}$ disconnect it.

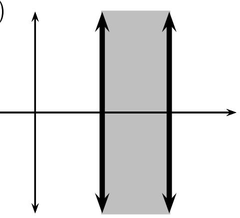
i)



ii)



iii)



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