# Homework Assignment #6

17.2 For each of the following functions defined on $\mathbb{R}^3$, find the critical points and classify them as local max, local min, saddle point or "can't tell":

$$
\begin{aligned}
&a)\ x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z,\\
&b)\ (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}
\end{aligned}
$$

**Answer:**

*a)* The first-order conditions are:

$$
\begin{aligned}
0 &= 2x + 6y - 10\\
0 &= 6x + 2y - 3z - 5\\
0 &= -3y + 8z - 21.
\end{aligned}
$$

It is easily verified that the solution is $(x, y, z) = (2, 1, 3)$ (one can use Cramer's rule or Gaussian elimination to find it). This is the only critical point. The Hessian is

$$
\mathbf{H} = \begin{pmatrix} 2 & 6 & 0\\ 6 & 2 & -3\\ 0 & -3 & 8 \end{pmatrix}.
$$

The first two leading principal minors of the Hessian are $\mathbf{H}_1 = 2 > 0$ and $\mathbf{H}_2 = -32 < 0$. This is indefinite, so the critical point is a saddlepoint.

*b)* To simplify notation, set $\psi = e^{-(x^2+y^2+z^2)}$. The first-order conditions are:

$$
\begin{aligned}
0 &= 2x[1 - x^2 - 2y^2 - 3z^2]\psi\\
0 &= 2y[2 - x^2 - 2y^2 - 3z^2]\psi\\
0 &= 2z[3 - x^2 - 2y^2 - 3z^2]\psi.
\end{aligned}
$$

At most one of $x$, $y$, and $z$ can be non-zero, otherwise the first-order conditions would give two different values for $x^2 + 2y^2 + 3z^2$. The critical points are $(0, 0, 1)$, $(0, 0, -1)$, $(0, 1, 0)$, $(0, -1, 0)$, $(1, 0, 0)$, $(-1, 0, 0)$, and $(0, 0, 0)$.

We now consider the Hessian. The second derivatives of our function (which we will call $f$) are:

$$
\begin{aligned}
f_{xx} &= [2(1 - 2x^2)(1 - x^2 - 2y^2 - 3z^2) - 4x^2]\psi\\
f_{yy} &= [2(1 - 2y^2)(2 - x^2 - 2y^2 - 3z^2) - 8x^2]\psi\\
f_{zz} &= [2(1 - 2z^2)(3 - x^2 - 2y^2 - 3z^2) - 12x^2]\psi\\
f_{xy} &= [-8xy - 4xy(1 - x^2 - 2y^2 - 3z^2)]\psi\\
f_{xy} &= [-12xz - 4xz(1 - x^2 - 2y^2 - 3z^2)]\psi\\
f_{xy} &= [-12xy - 4yz(2 - x^2 - 2y^2 - 3z^2)]\psi.
\end{aligned}
$$

Note that the cross partials are zero at every critical point. Moreover, only the squared values of the variables appear. Finally, $\psi(0,0,0) = 1$ and $\psi = 1/e$ at the other critical points. We therefore must consider four Hessians.

$$\mathbf{H}(\pm 1,0,0) = \frac{1}{e}\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$
$$\mathbf{H}(0,\pm 1,0) = \frac{1}{e}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
$$\mathbf{H}(0,0,\pm 1) = \frac{1}{e}\begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -12 \end{pmatrix}$$
$$\mathbf{H}(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

The first two pairs are indefinite, so the critical points $(\pm 1,0,0)$ and $(0,\pm 1,0)$ are saddlepoints. The last pair is negative definite, so $(0,0,\pm 1)$ are local (in fact, global) maxima. Finally, $\mathbf{H}(0,0,0)$ is positive definite, so $(0,0,0)$ is a local minimum.

17.6 Dingbat Airlines has regular flights between Ypsilanti and Kalamazoo. It can treat business and pleasure travelers as separate markets by demanding advance purchase and Saturday night stay-over for pleasure travelers. Suppose that it notes a demand function of $Q = 16 - p$ for business travelers and a demand function $Q = 10 - p$ for pleasure travelers and that it has a cost function for all travelers of $C(Q) = 10 + Q^2$. How much should it charge in each market to maximize its profit?

**Answer:** We solve this in terms of prices. It can also be solved in terms of quantities.

Let $p_b$ and $p_p$ be the prices for business and pleasure travelers, respectively. Treating them as separate markets, revenue is $p_b(16 - p_b) + p_p(10 - p_p)$, the quantity produced is $26 - p_b - p_b$ and cost is $10 + (26 - p_b - p_p)^2$. Profit is $\pi(p_b, p_p) = p_b(16 - p_b) + p_p(10 - p_p) - 10 - (26 - p_b - p_p)^2$. The first order conditions are

$$\frac{\partial \pi}{\partial p_b} = 68 - 4p_b - 2p_p = 0 \text{ and}$$
$$\frac{\partial \pi}{\partial p_p} = 62 - 2p_b - 4p_p = 0.$$

The Hessian is:

$$\mathbf{H} = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}.$$

The leading principal minors are $\mathbf{H}_1 = -4$ and $\mathbf{H}_2 = 12$, indicating that we have a maximum. The profit maximizing prices are $p_b = 37/3$ and $p_p = 28/3$ with corresponding quantities $q_b = 11/3$ and $q_p = 2/3$.

17.9

*a)* Prove that $2ab \leq a^2 + b^2$ for all numbers $a$, $b$.

**Answer:** Use $0 \leq (a-b)^2 = a^2 - 2ab + b^2$. Rearrange to complete the proof.

*b*) Use this result to show that

$$\begin{aligned}(x_1 + \cdots + x_n)^2 &= x_1^2 + \cdots + x_n + \sum_{i<j} 2x_i x_j \\ &\le x_1^2 + \cdots + x_n^2 + (n-1)(x_1^2 + \cdots + x_n^2) \\ &= n(x_1^2 + \cdots + x_n^2)\end{aligned}$$

**Answer:** The first equality comes from carrying out the multiplication. Now $2x_i x_j \le (x_i^2 + x_j^2)$. When we look at the sum $\sum_{i<j} 2x_i x_j \le \sum_{i<j} x_i^2 + x_j^2$, we find that each $x_k$ appears $(n-1)$ times, so $\sum_{i<j} x_i^2 + x_j^2 = (n-1)\sum_{i=1}^n x_i^2$, yielding line 2. Line follows immediately.

*c*) Conclude that the point $(m^*, b^*)$ in (14) and (15) is a global minimizer of the function $S$ in (11).

**Answer:** The Hessian of $S$ is

$$\mathbf{H} = \begin{pmatrix} 2n & 2\sum_i x_i \\ 2\sum_i x_i & 2\sum_i x_i^2 \end{pmatrix}.$$

Here $\mathbf{H}_1 = 2 > 0$ and $\mathbf{H}_2 = 4n[(\sum_i x_i^2) - (\sum_i x_i)^2]$. Unless $x_i = x_j = x$ for all $i, j$, $\mathbf{H}_2 > 0$. Note that in that case, the other first-order principal minor is non-negative. It follows that $\mathbf{H}$ is positive semidefinite for all $\mathbf{x} \in \mathbb{R}^n$ (and mostly positive definite). Therefore the solution to the first order conditions is a minimum.

18.2 Find the maximum and minimum distance from the origin to the ellipse $x^2 + xy + y^2 = 3$. [Hint: Use $x^2 + y^2$ as your objective function.]

**Answer:** The problem is to maximize (minimize) $x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 3$. Note that the constraint function has derivative $dh = (2x + y, x + 2y)$ which is non-zero on the ellipse $x^2 + xy + y^2 = 3$. This establishes constraint qualification.

We then form the Lagrangian $\mathcal{L} = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$, which yields first-order conditions

$$\begin{aligned}0 &= 2x - \lambda(2x + y) \\ 0 &= 2y - \lambda(x + 2y).\end{aligned}$$

We divide to eliminate $\lambda$, obtaining $x/y = (2x + y)/(2y + x)$. Clearing the fractions yields $x^2 = y^2$.

There are two cases: $x = y$ and $x = -y$. Substituting into the constraint, we find that the first has solution $x = \pm 1$ and the second has solution $x = \pm\sqrt{3}$. The resulting critical points are $\pm(1, 1)$ and $\pm(\sqrt{3}, -\sqrt{3})$. The first two minimize the distance ($\sqrt{2}$) and the second two maximize it ($\sqrt{6}$).

18.10 Find the maximizer of $f(x, y) = x^2 + y^2$, subject to the constraints $2x + y \le 2$, $x \ge 0$, $y \ge 0$.

**Answer:** We first consider constraint qualification. The derivative of the constraint matrix, once the equations are all in the proper form, is

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Each row is non-zero and any two rows are linearly independent. The NDCQ will be satisfied whenever one or two constraints bind. As at most two constraints can bind, the NDCQ is satisfied.

Now form the Lagrangian $\mathcal{L} = x^2 + y^2 - \lambda(2x + y - 2) + \mu_x x + \mu_y y$. The first order conditions are

$$\begin{aligned}0 &= 2x - 2\lambda + \mu_x \\ 0 &= 2y - \lambda + \mu_y.\end{aligned}$$

We rewrite these as $2x + \mu_x = 2\lambda$ and $2y + \mu_y = \lambda$.

It is usually helpful to check whether any multiplier must be positive. In fact, they need not as one solution is $(x, y) = (0, 0)$ with $\lambda = \mu_x = \mu_y = 0$.

However, if either $x > 0$ or $y > 0$, the first order conditions imply that $\lambda > 0$. Then by complementary slackness, $2x + y = 2$. This cuts the number of remaining cases down from seven to three. We consider them in turn.

A) If $y = 0$ and $x > 0$, $x = 1$. Here $(x, y) = (1, 0)$ and $\lambda = 1$, $\mu_x = 0$, and $\mu_y = 1$ is ok.

B) If $x = 0$ and $y > 0$, $y = 2$. Here $(x, y) = (0, 2)$ and $\lambda = 4$, $\mu_x = 8$, and $\mu_y = 0$ is ok.

C) Finally, if $x, y > 0$, both $\mu_x, \mu_y = 0$ so $2x = 2\lambda$ and $y = \lambda$. Then the constraint $2x + y = 2$ implies $(x, y) = (4/5, 2/5)$ and $\lambda = 4/5$.

The four critical points are $(0, 0)$, $(1, 0)$, $(0, 2)$, and $(4/5, 2/5)$. Calculating the values at the four critical points, we obtain $f(0, 0) = 0$, $f(1, 0) = 1$, $f(0, 2) = 4$ and $f(4/5, 2/5) = 4/5$. Thus $(x, y) = (0, 2)$ is the maximizer.

18.13 Show that the budget inequality constraint is binding in Example 18.8 even in the presence of the non-negativity constraints $x_1 \geq 0$, $x_2 \geq 0$. In the process, check the NDCQ for this more general problem.

**Answer:** We consider NDCQ first. The matrix formed from the derivatives of the constraint functions is:

$$\begin{pmatrix} p_1 & p_2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

At most two of the three constraints can bind simultaneously. With $p_i > 0$, the rank of any of the 3 matrices formed by deleting one row is 2, as required. The rank of any of the 3 matrices formed by deleting two rows is 1, as required. Thus NDCQ is satisfied.

We now turn to the Lagrangian. It is

$$\mathcal{L} = U(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - I) + \mu_1 x_1 + \mu_2 x_2.$$

The first-order conditions are $\partial U/\partial x_1 + \mu_1 = \lambda p_1$, $\partial U/\partial x_2 + \mu_2 = \lambda p_2$. If $\lambda = 0$, the first-order conditions become

$$\frac{\partial U}{\partial x_1} = -\mu_1 \leq 0$$
$$\frac{\partial U}{\partial x_2} = -\mu_2 \leq 0.$$

It is impossible to satisfy these equations because Example 18.8 assumes that $\partial U/\partial x_i > 0$, which implies $\mu_i < 0$, violating non-negativity. Therefore $\lambda > 0$ and complementary slackness implies that the budget constraint must bind.