Homework Assignment #7

19.3 If x thousand dollars is spent on labor and y thousand dollars is spent on equipment, a certain factory produces $Q(x, y) = 50x^{1/2}y^2$ units of output.

a) How should $80,000 be allocated between labor and equipment to yield the largest possible output?

b) Use Theorem 19.1 to estimate the change in maximum output if this allocation decreased by $1000.

c) Compute the exact change in b).

Answer:

a) We to maximize $Q = 50x^{1/2}y^2$ subject to the constraint that $x + y = 80$. The Lagrangian is $\mathcal{L} = 50x^{1/2}y^2 - \mu(x + y - 80)$. The resulting first order conditions are

$$\mu = 25x^{-1/2}y^2$$
$$\mu = 100x^{1/2}y.$$

Eliminating μ, we find that $y = 4x$. The solution is $x = 16$, $y = 64$, $\mu = 25600$. The value of output is $819,200.

b) By Theorem 19.1, the estimated change in the value of output is $\mu \times -1 = -25600$.

c) We must still spend in a 4-1 ratio, so $x = 15.8$, $y = 63.2$. Substituting in the production function, we find output is now worth $793,839. The actual change is $-25,361, slightly smaller than the approximation of $-25,600.

19.22 Consider the problem of maximizing x subject to the $y - x^4 \leq 0$, $x^3 - y \leq 0$, and $x \leq 1/2$. Try solving this problem with and without using a multiplier λ_0 for the objective function.

Answer: The Lagrangian without a multiplier on the objective is $\mathcal{L} = x - \lambda_1(y - x^4) - \lambda_2(x^3 - y) - \lambda_3(x - 1/2)$. The first order conditions are

$$\partial\mathcal{L}/\partial x = 1 + 4\lambda_1 x^3 - 3\lambda_2 x^2 - \lambda_3 = 0$$
$$\partial\mathcal{L}/\partial y = -\lambda_1 + \lambda_2 = 0,$$

The second condition implies $\lambda_1 = \lambda_2$. If either is positive, both are, and complementary slackness implies $x^4 = y = x^3$. This has solutions $x = 0$ and $x = 1$. The latter is infeasible ($x \leq 1/2$), so in this case $x = y = 0$. Then the other first order condition yields $\lambda_3 = 1$. At this point complementary slackness implies $x = 1/2$, which contradicts $x = 0$.

Thus $\lambda_1 = \lambda_2 = 0$. Again $\lambda_3 = 1$, so $x = 1/2$. But then $y \leq 1/16$ and $y \geq 1/8$, which is impossible. It follows that there are no solutions to the first order conditions.

However, if we put a multiplier on the objective, we obtain

$$\partial\mathcal{L}/\partial x = \lambda_0 + 4\lambda_1 x^3 - 3\lambda_2 x^2 - \lambda_3 = 0$$
$$\partial\mathcal{L}/\partial y = -\lambda_1 + \lambda_2 = 0,$$

Again $\lambda_2 = \lambda_3$, but the case $x = 0$ yields $\lambda_0 = \lambda_3$ and we can set $\lambda_0 = \lambda_3 = 0$, $\lambda_1 = \lambda_2 \geq 0$, and $x = y = 0$ to obtain a solution.

20.4 Consider the constant elasticity of substitution (CES) production function $F(x_1, x_2) = A(a_0 + a_1 x_1^\rho + a_2 x_2^\rho)^{1/\rho}$. Show that F has constant returns to scale when $a_0 = 0$.

Answer: When $a_0 = 0$, the production function is $F(x_1, x_2) = A(a_1 x_1^\rho + a_2 x_2^\rho)^{1/\rho}$. Then $F(\alpha x_1, \alpha x_2) = A[a_1(\alpha x_1)^\rho + a_2(\alpha x_2^\rho)]^{1/\rho} = A[\alpha^\rho(a_1 x_1^\rho + a_2 x_2^\rho)]^{1/\rho} = \alpha A[(a_1 x_1^\rho + a_2 x_2^\rho)]^{1/\rho} = \alpha F(x_1, x_2)$.

20.5 If $y = f(x_1, x_2)$ is C^2 and homogeneous of degree r, show that

$$x_1^2 f''_{x_1 x_1} + 2x_1 x_2 f''_{x_1 x_2} + x_2^2 f''_{x_2 x_2} = r(r-1)f.$$

Answer: Let $\varphi(t) = f(t\mathbf{x}) = t^r f(\mathbf{x})$. The first t-derivative is

$$\varphi'(t) = x_1 f'_{x_1}(t\mathbf{x}) + x_2 f'_{x_2}(t\mathbf{x}) = rt^(r-1)f(\mathbf{x}).$$

Differentiate again to obtain

$$\varphi''(t) = x_1(x_1 f''_{x_1 x_1} + x_2 f''_{x_1 x_2}) + x_2(x_1 f''_{x_1 x_2} + x_2 f''_{x_2 x_2}) = r(r-1)t^{r-2}f(\mathbf{x})$$

with the f'' terms evaluated at $t\mathbf{x}$. Now set $t = 1$ anbd collect terms to obtain the result.

Alternatively, one can use the fact that the derivative of an H.D. r function is H.D. $(r-1)$.

21.1 Prove that every linear function is homogeneous, concave, *and* convex. **Answer:** Recall that a function is linear whenever $f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$ for any real numbers α and β and any vectors $\mathbf{x}$ and $\mathbf{y}$.

Setting $\alpha = \beta = 0$, we find that $f(\mathbf{0}) = 0$. Then setting $\mathbf{y} = \mathbf{0}$, we obtain $f(\alpha\mathbf{x}) = \alpha f(\mathbf{x})$, proving f is homogeneous of degree 1.

Setting $\alpha = 1 - t$ and $\beta = t$, we find $f\big((1-t)\mathbf{x} + t\mathbf{y}\big) = (1-t)f(\mathbf{x}) + tf(\mathbf{y})$, which establishes that f is both concave and convex.

21.3 Prove that a quadratic form on $\Re^n$ is concave if and only if it is negative semidefinite. Prove that it is convex if and only if it is positive semidefinite. What can be said about the more general "quadratic function" $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c$?

Answer: Let $Q(x) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form where A is a symmetric matrix. The Hessian of Q is $2A$. Theorem 21.5 then applies, telling us that Q is convex if and only A is positive semidefinite, and that Q is concave if and only if A is negative semidefinite.

As far as the quadratic function $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c$ is concerned, its Hessian is also the matrix $2A$. Thus it is convex if and only if A is positive semidefinite, and concave if and only if A is negative semidefinite.