Homework Assignment #8

23.8 Find the general solution of the following systems of difference equations.

$$a)\ \begin{aligned} x_{n+1} &= 3x_n \\ y_{n+1} &= x_n + 2y_n; \end{aligned} \qquad b)\ \begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= -x_n + 5y_n; \end{aligned} \qquad c)\ \begin{aligned} x_{n+1} &= x_n - y_n \\ y_{n+1} &= 2x_n + 4y_n; \end{aligned}$$

$$d)\ \begin{aligned} x_{n+1} &= 3x_n - y_n \\ y_{n+1} &= -x_n + 2y_n - z_n; \\ z_{n+1} &= -y_n + 3z_n; \end{aligned} \qquad e)\ \begin{aligned} x_{n+1} &= 4x_n - 2y_n - 2z_n \\ y_{n+1} &= y_n; \\ z_{n+1} &= x_n + z_n. \end{aligned}$$

Answer:

a) We can write this as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

The characteristic equation is $(3-\lambda)(2-\lambda) = 0$, so the eigenvalues are $\sigma(A) = \{2, 3\}$. The vectors $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are corresponding eigenvectors. The general solution is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 2^n c_2 \mathbf{v}_2 + 3^n c_3 \mathbf{v}_3$ or

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_3 3^n \\ c_2 2^n + c_3 3^n \end{pmatrix}$$

b) We can write this as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

The characteristic equation is $\lambda^2 - 5\lambda + 1 = 0$, so the eigenvalues are $\sigma(A) = \{\frac{5+\sqrt{21}}{2}, \frac{5-\sqrt{21}}{2}\}$. The vectors $\mathbf{v}_+ = \begin{pmatrix} 1 \\ \frac{5+\sqrt{21}}{2} \end{pmatrix}$ and $\mathbf{v}_- = \begin{pmatrix} 1 \\ \frac{5-\sqrt{21}}{2} \end{pmatrix}$ are corresponding eigenvectors. The general solution is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (\frac{5+\sqrt{21}}{2})^n c_+ \mathbf{v}_+ + (\frac{5-\sqrt{21}}{2})^n c_- \mathbf{v}_-$.

c) We can write this as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

The characteristic equation is $\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$, so the eigenvalues are $\sigma(A) = \{2, 3\}$. The vectors $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_3 - = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are corresponding eigenvectors. The general solution is $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 2^n c_2 \mathbf{v}_2 + 3^n c_3 \mathbf{v}_3$.

d) Let

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

The characteristic equation is $0 = (3-\lambda)[(2-\lambda)(3-\lambda)-2] = (3-\lambda)(\lambda-4)(\lambda-1)$ so $\sigma(A) = \{1, 3, 4\}$. Corresponding eigenvectors are $\mathbf{v}_1 = (1, 2, 1)'$, $\mathbf{v}_3 = (1, 0, -1)'$, and $\mathbf{v}_4 = (-1, 1, -1)'$. The general solution is

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 3^n \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_4 4^n \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_3 3^n - c_4 4^n \\ 2c_1 + c_4 4^n \\ c_1 - c_3 3^n - c_4 4^n \end{pmatrix}$$

e) Let

$$A = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The characteristic equation is $(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$ which has solutions $\sigma(A) = \{1, 2, 3\}$. Corresponding eigenvectors are $\mathbf{v}_1 = (0, 1, -1)'$, $\mathbf{v}_2 = (1, 0, 1)'$, $\mathbf{v}_3 = (2, 0, 1)'$. The general solution is

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = c_1\mathbf{v}_1 + c_2 2^n \mathbf{v}_2 + c_3 3^n \mathbf{v}_3 = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_2 2^n \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 3^n \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

23.11 Suppose that the two-year species in Example 23.7 evolves to a three-year species with death rates $d_1 = 0.5$, $d_2 = 0.8$, $d_3 = 1$ and birth rates $b_1 = 1$, $b_2 = 4$, and $b_3 = 0$. Write out the corresponding Leslie system and find its general solution.

Answer: The model is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} x_n + 4y_n \\ 0.5x_n \\ 0.2y_n \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

It is easily verified that the matrix has eigenvalues $+2, -1, 0$ with corresponding eigenvectors

$$\begin{pmatrix} 4 \\ 1 \\ 0.1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -0.2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It follows that the general solution is

$$\mathbf{x}_n = \alpha 2^n \begin{pmatrix} 4 \\ 1 \\ 0.1 \end{pmatrix} + \beta(-1)^n \begin{pmatrix} -2 \\ 1 \\ -0.2 \end{pmatrix} + \gamma 0^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where we interpret $0^0 = 1$ and $0^n = 0$ for $n > 0$.

23.21

a) Show that the matrix $B = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ has 2 as an eigenvalue of multiplicity three, but only one independent eigenvector $\mathbf{v}_1$.

b) Use (31) to find generalized eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$.

c) If $P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, show that $P^{-1}BP$ has the desired form.

Answer:

a) The eigenvalues obey $(1-\lambda)(2-\lambda)(3-\lambda)+2-\lambda=0$. This can be rewritten as $(2-\lambda)^3=0$, showing that 2 is an eigenvalue of multiplicity 3. Then $B-2I=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$. If $(B-2I)\mathbf{v}=0$, the components of $\mathbf{v}$ obey $v_1+v_3=0$ and $v_1+v_2+v_3=0$. This means that $\mathbf{v}_1$ is a multiple of $(1,0,-1)^T$. We set $\mathbf{v}_1=(1,0,-1)^T$.

b) We next find a solution to $(B-2I)\mathbf{v}_2=\mathbf{v}_1$. Here $\mathbf{v}_2=(1,1,-1)^T$ will do. Then set $(B-2I)\mathbf{v}_3=\mathbf{v}_2$. The vector $\mathbf{v}_3=(1,0,0)^T$ solves the equation.

c) Then $P=[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$. This has inverse $P^{-1}=\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. It is then easily verified that $P^{-1}BP=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

23.38 For each of the following symmetric matrices A, find a matrix Q such that $Q^{-1}=Q^T$ and Q^TAQ is diagonal.

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Answer:

a) The eigenvalue equation is $0=(2-\lambda)^2-1=\lambda^2-4\lambda+3=(\lambda-3)(\lambda-1)$. The eigenvalues are $\lambda=1,3$. To find the eigenvectors, we set $(A-\lambda I)\mathbf{v}=\mathbf{0}$, which yields eigenvectors $\mathbf{v}_3=(1,1)'$ and $\mathbf{v}_1=(1,-1)'$. These are automatically orthogonal since A is symmetric and the eigenvalues are different. To find Q, we first normalize the eigenvectors before using them as the columns of Q. This yields

$$Q=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

b) The eigenvalue equation is $0=(1-\lambda)^2-9=\lambda^2-2\lambda-8=(\lambda-4)(\lambda+2)$. The eigenvalues are $\lambda=4,-2$. To find the eigenvectors, we set $(A-\lambda I)\mathbf{v}=\mathbf{0}$, which yields eigenvectors $\mathbf{v}_4=(1,1)'$ and $\mathbf{v}_{-2}=(1,-1)'$. These are again automatically orthogonal. To find Q, we first normalize the eigenvectors before using them as the columns of Q. This yields

$$Q=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

c) The eigenvalue equation is $0=(2-\lambda)[(2-\lambda)^2-1]-[2-\lambda+1]-[1+2-\lambda]=-\lambda(\lambda^2-6\lambda+9)=-\lambda(\lambda-3)^2$. The eigenvalues are $\lambda=3,3,0$. We first find an eigenvector for $\lambda=0$. We must solve

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\mathbf{v}=\mathbf{0}.$$

A solution is $\mathbf{v}_0=(1,1,1)'$. Next we must find eigenvectors for $\lambda=3$. The eigenvalue is repeated and we must find two orthogonal eigenvectors for $\lambda=3$. We solve

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\mathbf{v}=\mathbf{0}.$$

Obviously, there is only one independent equation, so the eigenvectors solve $v_1 + v_2 + v_3 = 0$. We have two free parameters. There are many ways to find independent eigenvectors. We start by picking a solution. E.g., $(1, -1, 0)'$. We then look for a second solution that is perpendicular to the first. It must satisfy both $v_1 + v_2 + v_3 = 0$ and $0 = \mathbf{v} \cdot (1, -1, 0)' = v_1 - v_2 = 0$. I.e., $v_1 = v_2$ and $2v_1 + v_3 = 0$. A solution is $(1, 1, -2)$. We now normalize before forming the matrix Q. We put the largest values first, so

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Other solutions also work.

24.14 Solve the following differential equations for initial values $y(0) = 2$, $\dot{y}(0) = 1$:

a) $\ddot{y} + 2\dot{y} + 10y = 0$;

b) $\ddot{y} + 9y = 0$.

Answer:

a) We first find the eigenvalues by substituting $y(t) = e^{rt}$. Then $(r^2+2r+10)e^{rt} = 0$, so $r^2+2r+10 = 0$. This has solutions $r = -1 \pm 3i$. Then the general solution has the form

$$y(t) = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t.$$

The derivative is

$$\dot{y}(t) = -c_1 e^{-1} \cos 3t - 3c_1 e^{-t} \sin 3t - c_2 e^{-t} \sin 3t + 3c_2 e^{-t} \cos 3t.$$

The initial conditions become $y(0) = c_1 = 2$ and $\dot{y}(0) = -c_1 + 3c_2 = 1$. It follows that $c_1 = 2$, $c_2 = 1$, and the solution is

$$y(t) = 2e^{-t} \cos 3t + e^{-t} \sin 3t.$$

b) We first find the eigenvalues by substituting $y(t) = e^{rt}$. Then $(r^2 + 9)e^{-rt} = 0$, so $r^2 + 9 = 0$. This has solutions $r = \pm 3i$ and the general solution has the form

$$y(t) = c_1 \cos 3t + c_2 \sin 3t.$$

The derivative is

$$\dot{y}(t) = -3c_1 \sin 3t + 3c_2 \cos 3t.$$

The initial conditions become $y(0) = c_1 = 2$ and $\dot{y}(0) = 3c_2 = 1$. The solution is

$$y(t) = 2 \cos 3t + \frac{1}{3} \sin 3t.$$

24.20 **(Principle of Superposition)** Show the if $y_{p_1}(t)$ is a solution of $a\ddot{y} + b\dot{y} + cy = g_1(t)$ and $y_{p_2}(t)$ is a solution of $a\ddot{y} + b\dot{y} + cy = g_2(t)$, then $y_{p_1}(t) + y_{p_2}(t)$ is a solution of $a\ddot{y} + b\dot{y} + cy = g_1(t) + g_2(t)$.

Answer: In this case

$$\frac{d}{dt}(y_{p_1} + y_{p_2}) = \dot{y}_{p_1} + \dot{y}_{p_2}$$

and

$$\frac{d^2}{dt^2}(y_{p_1} + y_{p_2}) = \ddot{y}_{p_1} + \ddot{y}_{p_2}.$$

Setting $y = y_{p_1} + y_{p_2}$, we find that $a\ddot{y} + b\dot{y} + cy = a\ddot{y}p_1 + b\dot{y}p_1 + cy_{p_1} + a\ddot{y}p_2 + b\dot{y}p_2 + cy_{p_2} = g_1 + g_2$.