Mathematical Economics Exam #2, November 7, 2019

1. Consider the problem of maximizing $u(x, y) = x - e^{-y}$ subject to the constraints $x, y \geq 0$ and $px + y \leq 10$ where $p > 0$.

   a) Is constraint qualification satisfied?

   Answer: The derivative of the constraints is:

$$\begin{pmatrix} p & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

   At most two of the constraints can bind. Since all rows are non-zero and any two rows are independent, the rank will equal the number of binding constraints. NDCQ is satisfied.

   b) Set up the Lagrangian for this problem and find the first-order conditions.

   Answer: The Lagrangian is $\mathcal{L} = x - e^{-y} - \lambda_0(px + y - 10) + \lambda_1 x + \lambda_2 y$. The first-order conditions are:

$$0 = 1 - p\lambda_0 + \lambda_1 \tag{1}$$

$$0 = e^{-y} - \lambda_0 + \lambda_2 \tag{2}$$

   c) Now find the solution to the maximization problem.

   Answer: Equation (1) tells us $p\lambda_0 = 1 + \lambda_1 \geq 1$. Then $\lambda_0 > 0$ and it follows that $px + y = 10$ by complementary slackness. The budget constraint always binds. Since all three constraints cannot simultaneously bind, this reduces our seven potential cases to three.

   The three cases are: A) $(x, y) = (0, 10)$; B) $(x, y) = (10/p, 0)$; C) $x, y > 0$.

   In case (A), $\lambda_2 = 0$ by complementary slackness, so $\lambda_0 = e^{-10}$. This requires $pe^{-10} \geq 1$, that is, $p \leq e^{10}$.

   In case (B), $\lambda_1 = 0$ by complementary slackness. Then $1 = p\lambda_0$ and $1 + \lambda_2 = \lambda_0$ by the first-order conditions. Combining them, we find $1 + \lambda_2 = 1/p$. Non-negativity will be satisfied if and only if $p \leq 1$.

   In case (C), $\lambda_1 = \lambda_2 = 0$ by complementary slackness. It follows that $p\lambda_0 = 1$. Then equation (2) becomes $e^{-y} = \lambda_0 = 1/p$, so $y = \ln p$. Since $y \geq 0$, this requires $p \geq 1$.

Moreover, $x = (10 - y)/p$, which must be non-negative. That happens if $y \ln p \le 10$. In other words, case (3) requires $1 \le p \le e^{10}$.

In sum, the solution is:

$$(x, y) = \begin{cases} (0, 10) & \text{for } e^{10} \le p \\ \left(\frac{10 - \ln p}{p}, \ln p\right) & \text{for } 1 \le p \le e^{10} \\ (10/p, 0) & \text{for } p \le 1. \end{cases}$$

Notice that the overlapping points are the same no matter which way we evaluate them.

2. Let $A = \{(x, y) : x^2 + y^2 \ge 1, x^2 + y^2 \le 5\}$.

   a) Is A closed? Explain.

   b) Is A bounded? Explain.

   c) Is A compact? Explain.

   d) Is A connected? Explain.

Answer:

a) **Yes**. The functions $f(x, y) = x^2 + y^2$ and $g(x, y) = 2x^2 + y^2$ are continuous. The set $A = f^{-1}[1, +\infty) \cap g^{-1}(-\infty, 5]$. Each of the two sets is closed as the inverse image of a closed set. Since A is the intersection of closed sets, it is closed.

b) **Yes**. Here $\|(x, y)\| \le \sqrt{5}$ for every $(x, y) \in A$, so the set is bounded.

c) **Yes**. By parts (a) and (b), A is closed and bounded, hence compact.

d) **Yes**. Perhaps the simplest way to see this is to write points in A in polar coordinates. They then have the form $r(\cos\theta, \sin\theta)$ where $1 \le r \le 5$ and $0 \le \theta \le 2\pi$. We can connect any $r_i(\cos\theta_i, \sin\theta_i)$ for $i = 1, 2$ by using the path $[r_1 + t(r_2 - r_1)]\big(\cos(\theta_1 + t(\theta_2 - \theta_1)), \sin(\theta_1 + t(\theta_2 - \theta_1))\big)$ for $t \in [0, 1]$. The path is in A because $1 \le r_1 + t(r_2 - r_1) \le 5$ Since A is path connected, it is connected.

3. Let $f(x, y) = xy^2 + x^3y - 2xy$. Find all critical points of f and classify them (local max, local min, saddlepoint, other/unknown).

Answer: The first-order conditions are

$$0 = y^2 + 3x^2y - 2y = y(y + 3x^2 - 2) \tag{1}$$

$$0 = 2xy + x^3 - 2x = x(2y + x^2 - 2) \tag{2}$$

From equation (1), we see that either $y = 0$ or $y + 3x^2 - 2 = 0$. In the $y = 0$ case, equation (2) yields $x = 0$ or $x = \pm\sqrt{2}$. The resulting critical points are $(0, 0)$, $(+\sqrt{2}, 0)$, and $(-\sqrt{2}, 0)$.

From equation (2), we see that either $x = 0$ or $2y + x^2 - 2 = 0$. If $x = 0$, we have $y = 2$ from equation (1). If $x \neq 0$, either $y = 0$ and we are in the previous case, or $y \neq 0$ and we are in the next case.

If neither $x = 0$ or $y = 0$, we have $y + 3x^2 - 2 = 0$ and $2y + x^2 - 2 = 0$. Eliminating y we find $x = \pm\sqrt{2/5}$. Substituting back, $y = 4/5$. Thus we have two more critical points: $(+\sqrt{2/5}, 4/5)$ and $(-\sqrt{2/5}, 4/5)$.

To sum up: $(0,0)$, $(0,2)$, $(+\sqrt{2},0)$, $(-\sqrt{2},0)$, $(+\sqrt{2/5},4/5)$, and $(-\sqrt{2/5},4/5)$ are the six critical points.

We now consider the Hessian

$$H = d^2f = \begin{pmatrix} 6xy & 2y + 3x^2 - 2 \\ 2y + 3x^2 - 2 & 2x \end{pmatrix}.$$

At $(0,0)$ and $(0,2)$, we find $H_1 = 0$ and $H_2 = -4$. These are **saddlepoints**. At $(\pm\sqrt{2},0)$, $H_1 = 0$ and $H_2 = -16$. Once again, we have **saddlepoints**.

That leaves $(\pm\sqrt{2/5},4/5)$. Here the Hessian becomes

$$H = d^2f = \begin{pmatrix} \pm\frac{24}{5}\sqrt{\frac{2}{5}} & 4/5 \\ 4/5 & \pm\sqrt{\frac{2}{5}} \end{pmatrix}.$$

Then $H_1 = \pm(24/5)\sqrt{2/5}$ and $H_2 = 32/25 > 0$. It follows that $(+\sqrt{2/5},4/5)$ is a **local minimum** and that $(-\sqrt{2/5},4/5)$ is a **local maximium**.

4. Consider the quadratic form $Q(x,y,z) = -x^2+xy+y^2+yz+z^2/2$ with constraint $x+y+z = 0$.

   a) Find a symmetric matrix that defines this quadratic form.

   Answer: The matrix that defines the quadratic form is

$$A = \begin{pmatrix} -1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

   b) Use the bordered Hessian to determine whether the quadratic form has a constrained maximum, minimum, or saddlepoint at $(0,0,0)$?

   Answer: We form the bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 1/2 & 0 \\ 1 & 1/2 & 1 & 1/2 \\ 1 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

There are 3 variables ($N = 3$) and one linear constraint ($M = 1$), so we look at the last $N - M = 2$ leading principal minors. They are $H_3 = +1$ and $H_4 = +1/2$. These minors are non-zero and have the same sign, so the matrix is either constrained positive definite or constrained indefinite. Moreover, $H_4 = 1/2$ is positive while both $(-1)^N$ and $(-1)^M$ are negative, so the form is constrained indefinite. It follows that there is a constrained saddlepoint at $(0, 0, 0)$.