# Homework Assignment #3

12.2 Explain why each of the following sets is not a subsequence of the last sequence in Example 12.2.

$$a)\ \left\{\frac{1}{1},\frac{3}{2},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\cdots\right\},\quad b)\ \left\{\frac{3}{1},\frac{3}{2},\frac{3}{3}\right\},\qquad c)\ \left\{\frac{1}{1},\frac{2}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\cdots\right\}.$$

**Answer:** The last sequence in Example 12.2 is

$$\frac{1}{1},\frac{3}{1},\frac{1}{2},\frac{3}{2},\frac{1}{3},\frac{3}{3},\frac{1}{4},\cdots.$$

Then $(a)$ is not a subsequence since $\frac{3}{2}$ must follow $\frac{1}{2}$; $(b)$ is not a subsequence because it is finite; $(c)$ is not a subsequence because $\frac{2}{1}$ is not in the sequence from Example 12.2.

12.7 Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers that converges to $x_0$ and that all $x_n$ and $x_0$ are nonzero.

*a)* Prove that there is a positive number $B$ such that $|x_n| \geq B$ for all $n$.

*b)* Using $a$, prove that $\{1/x_n\}$ converges to $1/x_0$.

**Answer:**

*a)* Since $x_n \to x_0$, we may choose $N$ so that $|x_n - x_0| < |x_0|/2$ for $n \geq N$. Then $|x_0| \leq |x_n| + |x_0 - x_n| \leq |x_n| + |x_0|/2$. Subtracting $|x_0|/2$ from both ends yields $|x_n| > |x_0|/2$ for $n \geq N$. Now let $B = \min\{|x_1|/2, |x_2|/2, \dots, |x_{N-1}|/2, |x_0|/2\}$. Since each $x_n$ and $x_0$ are nonzero, $B > 0$. For $n < N$, $|x_n| \geq 2B > B$ by construction while for $n \geq N$, $|x_n| > |x_0|/2 \geq B$ by our original choice of $N$. This establishes the result.

*b)* Let $\epsilon > 0$ be arbitrary. Choose $N$ such that $|x_n - x_0| < B|x_0|\epsilon$. Then, for $n \geq N$,

$$\begin{aligned}\left|\frac{1}{x_n} - \frac{1}{x_0}\right| &= \frac{1}{|x_n|\,|x_0|}|x_n - x_0| \\ &< \frac{1}{|x_n|\,|x_0|}B|x_0|\epsilon \\ &= \frac{B}{|x_n|}\epsilon < \epsilon\end{aligned}$$

where the last inequality follows from part (a). This establishes that $1/x_n \to 1/x_0$.

12.25 Show that the three examples in the last paragraph of section 12.4 are neither open nor closed.

**Answer:** The three examples are $(a, b]$ in $\mathbb{R}^1$, the sequence $\{1/n : n = 1, 2, \dots\}$ in $\mathbb{R}^1$, and a line minus a point in $\mathbb{R}^2$.

None of the examples are closed because each contains a sequence that converges to a limit outside the set. The sequence $a + (b - a)/2n$ is in $(a, b]$ and converges to $a \notin (a, b]$. The sequence $1/n$ converges to 0, which is not in the sequence. Any sequence in the line converging to the missing point does the trick in the third case.

None of the examples are open because in each case there is a point about which no balls are contained in the set. For any $\epsilon > 0$, $(b - \epsilon, b + \epsilon)$ contains points that are not in $(a, b]$. The interval $(2/3, 4/3)$ contains $1/1$, but no other points of the sequence $\{1/n\}$. Finally, a ball about any point of the line will contain points that are off to the side (recall that this line is in $\mathbb{R}^2$).

13.16 Suppose that $f$ and $g$ are both functions from $\mathbb{R}^k$ to $\mathbb{R}^1$, which are continuous at $\mathbf{x} \in \mathbb{R}^k$. Suppose that $g(\mathbf{x}) \neq 0$. Prove that the quotient function $f/g$ is defined and continuous at $\mathbf{x}$.

**Answer:** Let $\mathbf{x}_n \to \mathbf{x}$. Then $f(\mathbf{x}_n) \to f(\mathbf{x})$ and $g(\mathbf{x}_n) \to g(\mathbf{x})$ by continuity of $f$ and $g$. Since $g(\mathbf{x}) \neq 0$, Problem 12.8 applies to show that $\lim_{n\to\infty} f(\mathbf{x}_n)/g(\mathbf{x}_n) = f(\mathbf{x})/g(\mathbf{x})$. This establishes continuity of $f/g$ at $\mathbf{x}$.

29.4 Prove that a set of real number can have a most one least upper bound.

**Answer:** Suppose $a$ and $b$ are least upper bounds (suprema) of the non-empty set $A$. Then $a \leq b$ because both $a$ and $b$ are upper bounds of $A$ and $a$ is a least upper bound of $A$. Also, $b \leq a$ because both are upper bounds of $A$ and $b$ is a least upper bound. But then $a \leq b$ and $b \leq a$, so $a = b$.

29.13 Show that $N_{(a_1,\dots,a_n)}$ is a norm on $\mathbb{R}^n$ where $N_{(a_1,\dots,a_n)}(\mathbf{x}) = \|(a_1^{1/2}x_1, \dots, a_n^{1/2}x_n)\|_2$ is the weighted Euclidean norm on $\mathbb{R}^n$.

**Answer:** We must prove that $N_{\mathbf{a}}$ (1) is positive definite (2) is absolutely homogeneous of degree one and (3) obeys the triangle inequality.

By definition, $N_{\mathbf{a}}(\mathbf{x})$ is non-negative. If $N_{\mathbf{a}}(\mathbf{x}) = 0$, then each $a_i x_i^2 = 0$. Provided $\mathbf{a} \gg \mathbf{0}$, each $x_i = 0$, showing that $N_{\mathbf{a}}$ is **positive definite**. If any $a_i = 0$, $N_{\mathbf{a}}$ is not positive definite. The function $N_{\mathbf{a}}$ is still interpretable if some $a_i < 0$, but is not even positive in that case. For the rest of the problem, we assume $\mathbf{a} \gg \mathbf{0}$.

Now

$$N_{\mathbf{a}}(\lambda\mathbf{x}) = \left(\sum_{i=1}^{n} a_i \lambda^2 x_i^2\right) = |\lambda| \left(\sum_{i=1}^{n} a_i \lambda^2 x_i^2\right) = |\lambda| N_{\mathbf{a}}(\mathbf{x})$$

showing that $N_{\mathbf{a}}$ is absolutely homogeneous of degree one.

Let $f(\mathbf{x}) = (a_1^{1/2}x_1, \dots, a_n^{1/2}x_n)$. The function $f$ is linear on $\mathbb{R}^n$. Then Now $N_{\mathbf{a}}(\mathbf{x}) = \|f(\mathbf{x})\|_2$, so

$$N_{\mathbf{a}}(\mathbf{x}+\mathbf{y}) = \|f(\mathbf{x}+\mathbf{y})\|_2 = \|f(\mathbf{x}) + f(\mathbf{y})\|_2 \leq \|f(\mathbf{x})\|_2 + \|f(\mathbf{y})\|_2 = N_{\mathbf{a}}(\mathbf{x}) + N_{\mathbf{a}}(\mathbf{y})$$

proving the triangle inequality for $N_{\mathbf{a}}$.