Homework Assignment #1

14.6 Consider the constant elasticity demand function $Q = 6p_1^{-2}p_2^{3/2}$ where Q is the demand for good 1 and p_i is the price of good $i = 1, 2$. Suppose current prices are $p_1 = 6$ and $p_2 = 9$.

a) What is the current demand for Q?

b) Use differentials to estimate the change in demand as p_1 increases by 0.25 and p_2 decreases by 0.5.

c) Similarly, estimate the change in demand when both prices increase by 0.2.

d) Estimate the totals demands for situations b and c and compare your estimates with the actual demands.

Answer:

a) Demand is $Q = 4.5$.

b) Now $dQ = -12p_1^{-3}p_2^{3/2}dp_1 + 9p_1^{-2}p_2^{1/2}dp_2$. Evaluating at $(p_1, p_2) = (6, 9)$ we obtain $dQ = -(3/2)dp_1 + (3/4)dp_2$. Setting $dp_1 = 0.25$ and $dp_2 = -0.5$, we find $dQ = -3/8 - 3/8 = -3/4 = -0.75$.

c) Here $dQ = -.3 + 0.15 = -0.15$.

d) The actual values to 3 decimal places are $Q_b = 3.806$ and $Q_c = 4.356$, so $\Delta Q_b = -0.694$ vs. -0.75 and $\Delta Q_c = -0.144$ vs. -0.15.

14.12 At a given moment in time, the marginal product of labor is 2.5 and the marginal product of capital is 3, the amount of capital is increasing by 2 each unit of time and the rate of change of labor is +0.5. What is the rate of change of output?

Answer: The rate of change of output is $2.5 \times 0.5 + 3 \times 2 = 1.25 + 6 = 7.25$

14.17 Let $w(r, s)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^1$. Let $r = y - x$ and $s = y + x$. Let $F(x, y) = w\big(r(x, y), s(x, y)\big)$. Compute $\partial F/\partial x$ and $\partial F/\partial y$ in terms of $\partial w/\partial r$ and $\partial w/\partial s$.

Answer: We write (x, y) and (r, s) as column vectors. Then the Chain Rule yields:

$$\begin{aligned} dF = \left(\tfrac{\partial F}{\partial x} \quad \tfrac{\partial F}{\partial y} \right) &= dw \times d\begin{pmatrix} r \\ s \end{pmatrix} \\ &= \left(\tfrac{\partial w}{\partial r} \quad \tfrac{\partial w}{\partial s} \right) \times \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \left(-\tfrac{\partial w}{\partial r} + \tfrac{\partial w}{\partial s} \quad \tfrac{\partial w}{\partial r} + \tfrac{\partial w}{\partial s} \right). \end{aligned}$$

15.7 Consider the profit-maximizing firm in Example 15.5. If p increases by Δp and w increases by Δw, what will be the corresponding effect on the optimal input amount x?

Answer: Here $pf'(x) = w$ implicitly defines x as a function of (p, w). As long as $f'' \neq 0$, the implicit function theorem allows us to write $x = x(p, w)$. We write $F(x, p, w) = pf'(x) - w$, so that $F(x(p, w), p, w) = 0$. The implicit function theorem also tells us that $d_{(p,w)}x = -(1/pf''(x))d_{(p,w)}F = -(1/pf''(x))(f'(x), -1)$. The change in input is $\Delta x = d_{(p,w)}x(\Delta p, \Delta w)^T = (-1/pf''(x))(\Delta p f'(x) - \Delta w)$.

15.24 A firm uses two inputs to produce its output via the Cobb-Douglas production function $z = x^a y^b$, where $a = b = .5$. Its current level of inputs is $x = 25$, $y = 100$. The firm will introduce a new technology that will change the b-exponent on its production function to $b = .504$, with no change in a. Use calculus to estimate the input combination which will keep the total output the same and the sum of inputs the same.

Answer: Currently, $z = 50$. We want to find (x, y) that keeps $x^a y^b = 50$ (equivalently, $a \ln x + b \ln y = 50$, under the constraint that total input be unchanged, that $x + y = 125$. Define

$$F(x, y, b) = \begin{pmatrix} .5 \ln x + b \ln y - 50 \\ x + y - 125 \end{pmatrix}.$$

Our inputs $(x(b), y(b))$ must satisfy $F(x, y, b) = \mathbf{0}$. We wish to apply the implicit function theorem. We know $F(25, 100, .5) = \mathbf{0}$, and first examine $d_{(x,y)}F$.

$$d_{(x.y)}F = \begin{bmatrix} \frac{.5}{x} & \frac{b}{y} \\ 1 & 1 \end{bmatrix}.$$

Since $.5x - by \neq 0$, $d_{(x,y)}F$ is invertible. The implicit function theorem now tells us that

$$\begin{pmatrix} \partial x/\partial b \\ \partial y/\partial b \end{pmatrix} = -[d_{(x,y)}F]^{-1} \begin{bmatrix} \ln y \\ 0 \end{bmatrix}.$$

Thus

$$\begin{pmatrix} \partial x/\partial b \\ \partial y/\partial b \end{pmatrix} = \frac{-1}{ay - bx} \begin{bmatrix} xy & -bx \\ -xy & ay \end{bmatrix} [d_{(x,y)}F]^{-1} \begin{pmatrix} \ln y \\ 0 \end{pmatrix} = \frac{-1}{ay - bx} \begin{pmatrix} xy \ln y \\ -xy \ln y \end{pmatrix}.$$

Substituting in $(x, y, b) = (25, 100, .5)$ yields $\partial x/\partial b = -307 = -\partial y/\partial b$. Multiplying by $\Delta b = .004$, we find $\Delta x = -\Delta y = -1.228$.

29.12 For each of the following sets, decide whether or not the set is *a)* a subspace, *b)* closed, *c)* open, *d)* compact, *e)* connected.

$$i)\ \{(x_1, x_2) : x_1 = -x_2\}, \qquad ii)\ \{(x_1, x_2) : 1 < x_1^2 + x_2^2 < 2\},$$
$$iii)\ \{(x_1, x_2, x_3) : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}.$$

Answer:

i) This is a straight line through the origin. As a straight line, it **is a vector subspace**. Define $f(x_1, x_2) = x_1 + x_2$, then f is continuous and the set is $f^{-1}(\{0\})$, hence **closed**. It is clear that a ball about the origin cannot be contained in the set, so it is **not open**. It is not bounded, and so is **not compact**. Finally, as a straight line, it **is connected**.

ii) This is an open annulus. It is **not a vector subspace** as $(3/2, 0)$ is in the set, but $(3, 0)$ is not. It is **not closed** because $(2 - 1/n, 0)$ is in the set, but the limit $(2, 0)$ is not. It **is open** since it is the intersection of two open sets. Let f be the continuous function $f(x_1, x_2) = x_1^2 + x_2^2$ then the set is $f^{-1}(2, +\infty) \cap f^{-1}(-\infty, 1)$, which is open. It is not closed, so **not compact**. Finally, the set **is connected** as it is easy to make a path between any two points in it. E.g., from the starting point, continue around the set at the same distance from the origin, until you get to the angle of the end point, then head straight in or out to the end point.

iii) This is a budget set with prices $\mathbf{p} = (1, 1, 1)$ and income 1. It is **not a vector subspace** because it contains $(1, 0, 0)$ but not $(2, 0, 0)$. As a budget set, it is both **closed** and **compact**. It is **not open** as it does not contain any open balls about $(1, 0, 0)$. Finally, as a convex set, it is **connnected** (in fact, path connected).