Homework Assignment #5

15.22 Consider the system of equations

$$x + 2y + z = 5, \qquad 3x^2yz = 12,$$

as defining some endogenous variables in terms of some exogenous variables.

a) Divide the three variables into exogenous ones and endogenous ones in a neighborhood of $x = 2$, $y = 1$, $z = 1$ so that the Implicit Function Theorem applies.

Answer: Let F denote the left-hand side of the system. We calculate DF.

$$DF|_{(2,1,1)} = \begin{pmatrix} 1 & 2 & 1 \\ 6xyz & 3x^2z & 3x^2y \end{pmatrix}\Bigg|_{(2,1,1)} = \begin{pmatrix} 1 & 2 & 1 \\ 12 & 12 & 12 \end{pmatrix}$$

We must either treat (x, y) or (y, z) as endogenous to use the Implicit Function Theorem as the matrix will not be invertible if we remove the middle column, but is invertible if we remove either of the other columns.

We choose (x, y) as the endogenous variables and z as the exogenous variable.

b) If each of the exogenous variables in your answer to (a) increases by 0.24, use calculus to estimate how each of the endogenous variables will change.

Answer: The Implicit Function Theorem tells us that

$$\begin{pmatrix} \partial x/\partial z \\ \partial y/\partial z \end{pmatrix} = -\begin{pmatrix} 1 & 2 \\ 12 & 12 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 12 \end{pmatrix} = -\begin{pmatrix} -1 & \frac{1}{6} \\ 1 & -\frac{1}{12} \end{pmatrix} \begin{pmatrix} 1 \\ 12 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then $\Delta x \approx -1(.24) = -.24$ and $\Delta y \approx 0$.

15.36 Show that the map $F(x, y) = (x + e^y, y + e^{-x})$ is everywhere locally invertible.

Answer: Assuming F is a column vector, we compute

$$DF = \begin{bmatrix} 1 & e^y \\ -e^{-x} & 1 \end{bmatrix}.$$

The determinant is $1 + e^{y-x} > 1 > 0$. Since DF is always invertible, the Inverse Function Theorem guarantees the existence of a local inverse about any point (x_0, y_0).

16.2 Let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form on $\mathbb{R}^n$. By evaluating Q on each of the coordinate axes in $\mathbb{R}^n$, prove that a necessary condition for a symmetric matrix to be positive definite (positive semidefinite) is that all the diagonal entries be positive (nonnegative). State and prove the corresponding result for negative and negative semidefinite matrices. Give an example to show that this necessary condition is not sufficient.

Answer: The corresponding result is the following theorem.

THEOREM 1. *A necessary condition for for a symmetric matrix to be negative definite (negative semidefinite) is that all the diagonal entries are negative (nonpositive).*

PROOF. Calculate $Q(\mathbf{e}^i) = a_{ii}$. This must be negative for all i if Q is negative definite, nonpositive for all i if Q is negative semidefinite.

To show the necessary conditions are not sufficient, consider

$$A_1 = \begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}.$$

Then A_1 satisifes the necessary conditions for negative definiteness, and A_2 satisfies the necessary conditions for negative semidefiniteness. But the corresponding quadratic forms take the values $Q_1(1,1) = 6$ and $Q_2(1,1) = 8$, showing that A_1 is not negative definite and that A_2 is not negative semidefinite.

16.3 Using the method of the previous exercise, sketch a proof that if A is positive (or negative) definite, then every principal submatrix of A is also positive (or negative) definite.

Answer: Suppose $\mathbf{A}$ is positive definite and $\mathbf{A}_k$ is a principal submatrix of order k. Let $i_1, \ldots, i_k$ be the rows and columns included in $\mathbf{A}_k$.

Given $\mathbf{x} \in \mathbb{R}^k$, define $\hat{\mathbf{x}}$ by

$$\hat{x}_i = \begin{cases} x_i & \text{if } i \in \{i_1, \ldots, i_k\} \\ 0 & \text{if } i \notin \{i_1, \ldots, i_k\} \end{cases}$$

Then $\hat{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\hat{\mathbf{x}} = \mathbf{x}^{\mathbf{T}}\mathbf{A}_k\mathbf{x}$. Since $\mathbf{A}$ is positive, $\mathbf{A}_k$ is also positive. If $\mathbf{A}$ is definite, $\mathbf{x}^{\mathbf{T}}\mathbf{A}_k\mathbf{x} = 0$ implies $\hat{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\hat{\mathbf{x}} = 0$, so $\hat{\mathbf{x}} = \mathbf{0}$. It then follows that $\mathbf{x} = \mathbf{0}$. The negative case is similar.

30.7 Compute the Taylor polynomials of order one, two, and three of the function $y = \sqrt{x+1}$ about $x = 0$ and of the function $y = \ln x$ about $x = 1$. Then, compute the values of these approximations at $h = 0.2$ and $h = 1$, in each case comparing these approximations with the actual values

Answer: The derivatives of $f(x) = \sqrt{x+1}$ are

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}, \quad f''(x) = -\frac{1}{4}(x+1)^{-3/2}, \quad f'''(x) = \frac{3}{8}(x+1)^{-5/2}.$$

Then $f(0) = 1$, $f'(0) = 1/2$, $f''(0) = -1/4$, and $f'''(0) = 3/8$. The corresponding Taylor polynomials at $x = 0$ are:

$$P_1(x) = 1 + \frac{1}{2}x$$
$$P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$
$$P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

The derivatives of $f(x) = \ln x$ are $f'(x) = 1/x$, $f''(x) = -1/x^2$ and $f'''(x) = 2/x^3$. Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, and $f'''(1) = 2$. The corresponding Taylor polynomials at $x = 1$ are

$$P_1(x) = (x-1)$$
$$P_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$
$$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

For $f(x) = \sqrt{x+1}$. When $h = 0.2$, $x = 0.2$ and $P_1(.2) = 1.1$, $P_2(.2) = 1.095$, and $P_3(.2) = 1.0955$. These compare to $\sqrt{1.2} \approx 1.09545$.

When $h = 1$, $x = 1$ and we have $P_1(1) = 1.5$, $P_2(1) = 1.375$, and $P_3(1) = 1.4375$, compared to $\sqrt{2} \approx 1.4142$. Convergence is slower here.

For $f(x) = \ln x$, when $h = 0.2$, $x = 1.2$ and $P_1(1.2) = 0.2$, $P_2(.2) = 0.18$, and $P_3(1.2) \approx 0.1827$. These compare to $\ln 1.2 \approx 0.1823$.

When $h = 1$, $x = 2$ and we have $P_1(2) = 1$, $P_2(2) = 0.5$, $P_3(2) \approx 0.833$. These compare to $\ln 2 = 0.693$. Convergence is quite a bit slower here.