Homework Assignment #6

17.4 A firm uses two inputs to produce a single product. If its production function is $Q = x^{1/4}y^{1/4}$ and if it sells its output for a dollar a unit and buys each input for \$4 dollars a unit, find its profit-maximizing input bundle. (Check the second order conditions.)

Answer: Under these conditions, profit is $\pi(x, y) = x^{1/4}y^{1/4} - 4x - 4y$. We treat this as a unconstrained profit maximization problem since neither $x = 0$ nor $y = 0$ can yield positive profit. The first order conditions are $(1/4)x^{-3/4}y^{1/4} = 4$ and $(1/4)x^{1/4}y^{-3/4} = 4$. Dividing the first by the second yields $y/x = 1$, so $x = y$. Then substitute back in the production function to find $(1/4)x^{-1/2} = 4$. The solution is $x = y = 4^{-4} = 1/256$, which yields output $Q = 1/4$.

The Hessian of the objective function is:

$$\mathbf{H} = \frac{1}{16}\begin{pmatrix} -3x^{-7/4}y^{1/4} & x^{-3/4}y^{-3/4} \\ x^{-3/4}y^{-3/4} & -3x^{1/4}y^{-7/4} \end{pmatrix}.$$

At $x = y = 1/256$, this becomes

$$\mathbf{H} = 256\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}.$$

The leading principal minors are $\mathbf{H}_1 = -3(256) < 0$ and $\mathbf{H}_2 = (256)(9-1) > 0$, indicating that $\mathbf{H}$ is negative definite. This means $(x, y) = (1/256, 1/256)$ is a maximum.

17.7 For the discriminating monopolist of Example 17.3, compute the demand function for the market as a whole, without price discrimination. Compute the firm's profit-maximizing output for this situation and compare the profit to the computation in Example 17.3.

Answer: In Example 17.3: Type one consumers have demand price $50 - 5Q_1$ for $0 \le Q_1 \le 10$ and type two consumers have demand price $100 - 10Q_2$ for $0 \le Q_1 \le 10$. The cost of producing $Q = Q_1 + Q_2$ is $C(Q) = 90 + 20Q$. Example 17.3 found that profit was maximized when $(Q_1, Q_2) = (3, 4)$, so $Q = 7$. This resulted in prices $P_1 = 35$ and $P_2 = 60$. Profit was $35(3) + 60(4) - (90 + 140) = 115$.

Without price discrimination, the price is the same (p) in both markets. Quantity demanded obeys $p = 50 - 5Q_1$ in market 1 and $p = 100 - 10Q_2$ in market 2, provided $Q_i \ge 0$. The respective choke prices are $P_1 = 50$ and $P_2 = 100$.

There are two cases to consider, whether the price is above or below 50. If the price is above 50, the product will only be bought by consumers in market two. Then profit is $(100 - 10Q_2)Q_2 - (90 + 20Q_2) = -10Q_2^2 + 80Q_2 - 90$. The first-order condition for profit maximization is $-20Q_2 + 80 = 0$, so $Q_2 = 4$ and $p = 60$ (consistent with our assumption that $p > 50$). The second derivative is negative, so this is the maximization point. Computing profit, we find profit is $60 \times 4 - (90 + 20 \times 4) = 240 - 170 = 70$.

The second case has $p \le 50$. In that case we must add the demand curves **horizontally**. To do that we solve for quantity as a function of price. Thus $Q_1 = 10 - p/5$ and $Q_2 = 10 - p/10$ for $p \le P_2 = 100$. Total quantity demanded is $Q = Q_1 + Q_2 = 20 - 3p/10$.

Profit is then $p(Q_1 + Q_2) - (90 + 20(Q_1 + Q_2)) = (p - 20)(20 - 3p/10) - 90 = -3p^2/10 + 26p - 490$. The first-order conditions are $26 - 6p/10 = 0$, or $p = 130/3$. Since the second derivative is negative, this is a maximum. Note that $p = 130/3$ is less than 50, so case two has a solution. Now $Q_1 = 4/3$ and $Q_2 = 17/3$, so $Q = 7$. It follows that profit is $(130/3) \times 7 - (90 + 20 \times 7) = 910/3 - 230 = 220/3 = 73\frac{1}{3}$.

In this case it is better to sell both markets as the profit will be higher ($73\frac{1}{3}$ versus 70).

Compared to the example, the firm is selling the same quantity as the price discriminating monopolist, so costs are the same. However, revenue is lower. The single-price monopolist earns $130/3 \times 7 = 910/3 = 303\frac{1}{3}$ in revenue, while the price discriminating monopolist earns $35 \times 3 + 60 \times 4 = 345$. This translates into a profit of \$115, which is higher than the single price case (\$73.33).

18.5 Find the point closest to the origin in $\Re^3$ that is on both the planes $3x + y + z = 5$ and $x + y + z = 1$.

Answer: Maximize $x^2 + y^2 + z^2$ under the constraints $3x + y + z = 5$ and $x + y + z = 1$. The derivative of the constraints is

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

This clearly has rank 2, so constraint qualification is satisified.

The Lagrangian is $L = x^2 + y^2 + z^2 - \mu(3x + y + z - 5) - \nu(x + y + z - 1)$. The first-order conditions are $0 = 2x - 3\mu - \nu$, $0 = 2y - \mu - \nu$, and $0 = 2z - \mu - \nu$. Note that $y = z$. Thus the constraints become $3x + 2y = 5$ and $x + 2y = 1$. Clearly $x = 2$ and $y = z = -1/2$, which yields $\mu = 9/4$ and $\nu = -11/4$. The closest point is $(2, -1/2, -1/2)$.

18.9 Maximize $x^2y^2z^2$ subject to $x^2 + y^2 + z^2 = c^2$, where c is some fixed positive constant. What is the maximum value of the objective function on the constraint set? Show that for all x, y, z.

$$x^2y^2z^2 \leq \left(\frac{1}{3}(x^2 + y^2 + z^2)\right)^3, \text{ or } (x^2y^2z^2)^{1/3} \leq \frac{x^2 + y^2 + z^2}{3}.$$

Answer: The maximum value is positive (in fact, at least $(c/\sqrt{3})^6$ since $\frac{1}{\sqrt{3}}(c, c, c)$ satisfies the constraint. This requires that x, y, $z > 0$ at the maximum. The NDCQ conditions is satisfied at such points since $dg = (2x, 2y, 2z)$ where g is the constraint function.

The Lagrangian is $\mathcal{L} = x^2y^2z^2 - \lambda(x^2 + y^2 + z^2 - c^2)$. The first-order conditions are $2xy^2z^2 = 2\lambda x$, $2x^2yz^2 = 2\lambda y$, and $2x^2y^2z = 2\lambda z$.

Since the maximum is not at zero, we can divide the first-order conditions by x, y, and z, respectively. Then $y^2z^2 = x^2z^2 = x^2y^2 = \lambda$. It follows that $x^2 = y^2 = z^2$. Any (x, y, z) with $|x| = |y| = |z| = c/\sqrt{3}$ yields a maximum, and the maximum value is $c^6/27$.

The second-order conditions should also be checked, but this problem is from Chapter 18, so it is not required for credit.

Normalize (x, y, z), obtaining $(x^2 + y^2 + z^2)^{-1/2}(x, y, z)$. This satisfies the constraint for $c = 1$, so $\left((x^2 + y^2 + z^2)^{-1/2}\right)^6 x^2y^2z^2 \leq 1$. This implies

$$x^2y^2z^2 \leq \left(\frac{1}{3}(x^2 + y^2 + z^2)\right)^3, \text{ or } (x^2y^2z^2)^{1/3} \leq \frac{x^2 + y^2 + z^2}{3}.$$

18.11 Find the maximizer of $f(x, y) = 2y^2 - x$, subject to the constraints $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

Answer: We first form the Lagrangian $\mathcal{L} = 2y^2 - x - \lambda(x^2 + y^2 - 1) + \mu_x x + \mu_y y$. The first order conditions are

$$0 = -1 - 2\lambda x + \mu_x \tag{1}$$

$$0 = 4y - 2\lambda y + \mu_y. \tag{2}$$

We can rewrite equation (1) as $1 + 2\lambda x = \mu_x$.

We now check to see if any multipliers must be positive. If successful, it will allow us to reduce the number of cases to consider. If we fail, there will be $2^3 = 8$ cases. Since $\lambda \geq 0$, equation (1) becomes $\mu_x > 0$. Complementary slackness implies $x = 0$. As a result, there are now two cases to consider: $y = 0$ and $y > 0$.

If $y = 0$, $x^2 + y^2 = 0 < 2$, so $\lambda = 0$ by complementary slackness. We then have $\mu_x = 1$ and $\mu_y = 0$. Thus $(0, 0)$ is a critical point. Note that $f(0, 0) = 0$.

If $y > 0$, $\mu_y = 0$ by complementary slackness. Then equation (2) becomes $4y - 2\lambda y = 0$. Since $y > 0$, $\lambda = 2$. By complementary slackness, $x^2 + y^2 = 1$. Since $x = 0$, $y = 1$. This is also a critical point.

Finally, $f(0, 1) = 2 > f(0, 0) = 0$, so the maximum is at $(0, 1)$.

18.15 Maximize $3xy - x^3$ subject to the constraints $2x - y = -5$, $5x + 2y \geq 37$, $x \geq 0$, $y \geq 0$.

Answer: We start by considering the constraints. One of them is redundant. The equality constraint tells us that $y = 2x + 5$. Substituting in the first inequality constraint, we find $9x + 10 \geq 37$, so $x \geq 3 > 0$. The third constraint is redundant. Further, since $y = 2x + 5 \geq 11$, the fourth constraint is also redundant.

There are now two ways to solve the problem. The first is to substitute $2x + 5$ for y. In that case, we must maximize $g(x) = 6x^2 + 15x - x^3$ subject to the constraint $x \geq 3$. Now $g'(3) = 24 > 0$, so 3 is not a maximum. g eventually becomes negative as x increases, so we check the critical points, setting $g' = 12x + 15 - 3x^2 = 0$. Now $3x^2 - 12x - 5 = 3(x + 1)(x - 5)$, so $x = 5$ is the only critical point with $x \geq 3$, which must be the maximum. Thus $(x, y) = (5, 15)$ is the maximum for the original problem.

Method two is to use the Lagrangian

$$L = 3xy - x^3 - \mu(2x - y + 5) - \lambda_1(5x + 2y - 37) - \lambda_2 y.$$

The first-order conditions are

$$\begin{aligned} 0 &= 3y - 3x^2 - 2\mu - 5\lambda_1 \\ 0 &= 3x - \mu - 2\lambda_1. \end{aligned}$$

The derivative of the possibly binding constraints is

$$\begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}$$

which has rank 2. Constraint qualification will be satisfied whether or not the second constraint binds.

We first consider the case where the second constraint binds: $5x + 2y = 37$. Using the constraint $2x - y = -5$, we find that $(x, y) = (3, 11)$. The first-order conditions become

$$\begin{aligned} 0 &= 6 + 5\lambda_1 - 2\mu \\ 0 &= 9 + 2\lambda_1 + \mu. \end{aligned}$$

This yields $\lambda_1 = -8/3$, violating non-negativity.

Then $\lambda_1 = 0$ by complementary slackness and the first-order conditions are:

$$\begin{aligned} 0 &= 3y - 3x^2 - 2\mu \\ 0 &= 3x - \mu. \end{aligned}$$

Using $y = 2x + 5$, we find $3x^2 - 12x - 15 = 0$ as before, yielding a maximum at $(x, y) = (5, 15)$ with $\lambda_2 = 0$.