Homework Assignment #7

19.14 Check the second-order conditions for the solutions of the first-order conditions in Exercises 18.2, 18.3, 18.5, 18.6, and 18.7.

Answer:

18.2 The bordered Hessian of the Lagrangian is:

$$\begin{pmatrix} 0 & 2x+y & 2y+x \\ 2x+y & 2+2\mu & \mu \\ 2y+x & \mu & 2+2\mu \end{pmatrix}.$$

There is one constraint ($k = 1$) and two variables ($n = 2$), so we look only at the determinant of the full bordered Hessian.

When $\mu = -2$ and $y = -x$, the determinant is $8x^2 > 0$. Since $(-1)^n H_3 > 0$, the solutions $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ are local maxima.

When $\mu = -2/3$ and $y = x$, the determinant is $-24x^2 < 0$. Since $(-1)^k H_3 > 0$, the solutions $(1, 1)$ and $(-1, -1)$ are local minima.

18.3 The bordered Hessian of the Lagrangian is:

$$\begin{pmatrix} 0 & 2x & 1 \\ 2x & 2-2\mu & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

There is one constraint ($k = 1$) and two variables ($n = 2$), so we look only at the determinant of the full bordered Hessian.

The determinant is $2\mu - 2 - 8x^2$. Plugging in $x = 1.17$ and $\mu = -0.7$, $H_3 < 0$. This is the same sign as $(-1)^k$, so we have a local minimum.

18.5 The unbordered Hessian of the Lagrangian is:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This is a positive definite matrix, and so we have a local minimum.

18.6 The bordered Hessian of the Lagrangian is:

$$\begin{pmatrix} 0 & 0 & 2x & 2y & 2z \\ 0 & 0 & 0 & 1 & 0 \\ 2x & 0 & -2\mu & 0 & 0 \\ 2y & 1 & 0 & -2\mu & 0 \\ 2z & 0 & 0 & 0 & 2-2\mu \end{pmatrix}.$$

Here there are 3 variables ($n = 3$) and 2 constraints ($k = 2$). We again look only at the determinant of the full bordered Hessian ($3 - 2 = 1$).

After some tedious calculations, we find $H_5 = 8(1-\mu)x^2 - 8\mu z^2$.

When $\mu = -1$, $H_5 = 16x^2 + 8z^2 > 0$. Since $(-1)^k > 0$, this is a local minimum (in fact $(-1, 0, 0)$ is the global minimum).

When $\mu = 1/2$, $x = 1$, and $z = 0$, so $H_5 = 4 > 0$, which implies $(1, 0, 0)$ is also a local minimum.

When $\mu = +1$, $z = \pm\sqrt{3}/2$, so $H_5 = -8z^2 < 0$.

Since $(-1)^n < 0$, the points $(1/2, 0, \pm\sqrt{3}/2)$ are local maxima (in fact, these are global maxima).

18.7 The bordered Hessian of the Lagrangian is:

$$\begin{pmatrix} 0 & 0 & 0 & 2y & 2z \\ 0 & 0 & z & 0 & x \\ 0 & z & 0 & 0 & 1-\nu \\ 2y & 0 & 0 & -2\mu & 1 \\ 2z & x & 1-\nu & 1 & -2\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2y & 2z \\ 0 & 0 & z & 0 & x \\ 0 & z & 0 & 0 & 0 \\ 2y & 0 & 0 & -2\mu & 1 \\ 2z & x & 0 & 1 & -2\mu \end{pmatrix}.$$

I have simplified using $\nu = 1$.

As in 18.6, we look only at the determinant of the full bordered Hessian.

After some calculation, we find $H_5 = -8z^2[yz + \mu z^2 + \mu y^2]$ Since $z \neq 0$, we need only consider the sign of $-[yz + \mu z^2 + \mu y^2]$. Moreover, $y^2 + z^2 = 1$, so we consider the sign of $-yz - \mu$.

When $\mu = 1/2$, $y = z = \pm 1/\sqrt{2}$ and the Hessian is negative. Since $(-1)^n < 0$, the points $\pm(3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ are local maxima.

When $\mu = -1/2$, $-y = z = \pm/\sqrt{2}$ and $H_5 > 0$. Since $(-1)^k > 0$, the points $\pm(3\sqrt{2}, -1/\sqrt{2}, 1/\sqrt{2})$ are local minima.

19.18 Consider the problem of maximizing $x_1^2 x_2$ on the constraint set $2x_1^2 + x_2^2 = a$, as in Example 18.5. Use the Implicit Function Theorem directly on the first order conditions of this problem to prove that the solutions $x_1(a)$, $x_2(a)$, $\lambda(a)$ depend smoothly on the parameter a near $a = 3$.

Answer: The Lagrangian is $\mathcal{L} = x_1^2 x_2 - \lambda(2x_1^2 + x_2^2 - a)$. Note that $x_1^2 x_2 > 0$ is possible, so $x_1 \neq 0$ and $x_2 \neq 0$ at the maximum. This allows us to divide by x_1 or x_2. The first-order conditions and constraint can be simplified to:

$$\begin{aligned} 0 &= x_2 - 2\lambda \\ 0 &= x_1^2 - 2\lambda x_2 \\ 0 &= 2x_1^2 + x_2^2 - a. \end{aligned}$$

We have three equations and three unknowns (x_1, x_2, λ). In order to apply the Implicit Function Theorem, the derivative of this system with respect to (x_1, x_2, λ) must be non-singular. The derivative is:

$$\begin{pmatrix} 0 & 1 & -2 \\ 2x_1 & -2\lambda & -x_2 \\ 4x_1 & 4x_2 & 0 \end{pmatrix}$$

This has determinant $-4x_1(5x_2 + 4\lambda)$. When (x_1, x_2, λ) solve the equations, $2\lambda = x_2$ and the determinant reduces to $-40x_1x_2$, which is not zero. The Implicit Function Theorem then tells us that $(x_1(a), x_2(a), \lambda(a))$ are C^1 functions for any $a > 0$, in particular for a near 3.

19.22 Consider the problem of maximizing x subject to the $y - x^4 \leq 0$, $x^3 - y \leq 0$, and $x \leq 1/2$. Try solving this problem with and without using a multiplier λ_0 for the objective function.

Answer: The Lagrangian without a multiplier on the objective is $\mathcal{L} = x - \lambda_1(y - x^4) - \lambda_2(x^3 - y) - \lambda_3(x - 1/2)$. The first order conditions are

$$\begin{aligned} \partial\mathcal{L}/\partial x &= 1 + 4\lambda_1 x^3 - 3\lambda_2 x^2 - \lambda_3 = 0 \\ \partial\mathcal{L}/\partial y &= -\lambda_1 + \lambda_2 = 0, \end{aligned}$$

The second condition implies $\lambda_1 = \lambda_2$. If either is positive, both are, and complementary slackness implies $x^4 = y = x^3$. This has solutions $x = 0$ and $x = 1$. The latter is infeasible ($x \leq 1/2$), so in this case $x = y = 0$. Then the other first order condition yields $\lambda_3 = 1$. At this point complementary slackness implies $x = 1/2$, which contradicts $x = 0$.

Thus $\lambda_1 = \lambda_2 = 0$. Again $\lambda_3 = 1$, so $x = 1/2$. But then $y \leq 1/16$ and $y \geq 1/8$, which is impossible. It follows that there are no solutions to the first order conditions.

However, if we put a multiplier on the objective, we obtain

$$\begin{aligned}\partial\mathcal{L}/\partial x &= \lambda_0 + 4\lambda_1 x^3 - 3\lambda_2 x^2 - \lambda_3 = 0\\ \partial\mathcal{L}/\partial y &= -\lambda_1 + \lambda_2 = 0,\end{aligned}$$

Again $\lambda_2 = \lambda_3$, but the case $x = 0$ yields $\lambda_0 = \lambda_3$ and we can set $\lambda_0 = \lambda_3 = 0$, $\lambda_1 = \lambda_2 \geq 0$, and $x = y = 0$ to obtain a solution.

20.1 Which of the following functions are homogeneous? What are the degrees of homogeneity of the homogeneous ones?

$$\begin{aligned} &a)\ 3x^5y + 2x^2y^4 - 3x^3y^3, && b)\ 3x^5 + 2x^2y^4 - 3x^3y^4,\\ &c)\ x^{1/2}y^{-1/2} + 3xy^{-1} + 7, && d)\ x^{3/4}y^{1/4} + 6x,\\ &e)\ x^{3/4} + 6x + 4, && f)\ \frac{(x^2 - y^2)}{(x^2 + y^2)} + 3.\end{aligned}$$

Answer: Function (a) is homogeneous of degree 6. Function (b) is not homogeneous as the first 2 terms are h.d. 6 and the third is h.d. 7. Function (c) is homogeneous of degree 0. Function (d) is homogeneous of degree 1. Function (e) is not homogeneous due to the constant term. Function (f) is homogeneous of degree 0.

20.11 Which of the following are monotonic transformations of $\mathbb{R}_+$?

$$a)\ z^4 + z^2. \qquad b)\ z^4 - z^2, \qquad c)\ z/(z + 1), \qquad d)\ \sqrt{z}, \qquad e)\ \sqrt{z_2 + 4}$$

Answer: The functions in (a), (c), (d), and (e) are monotonic. Their derivatives are $4z^3 + 2z$, $(z+1)^{-2}$, $1/2\sqrt{z}$, and $z/\sqrt{z^2 + 4}$, which are all non-negative on $\mathbb{R}_+$. The function (b) is not monotonic. Its derivative is $4z^3 - 2z$ which can be positive ($z = 1$) or negative ($z = 1/4$).

20.18 Use Theorems 20.9 and 20.10 to check the homotheticity of the functions in Exercise 20.17 and to determine whether or not $f(x, y) = x^4 + x^2y^2 + y^4 - 3x - 8y$ is homothetic.

Answer: The functions in Exercise 20.17 are a) $e^{x^2y}e^{xy^2}$, b) $2\log x + 3\log y$, c) $x^3y^6 + 3x^2y^4 + 6xy^2 + 9$, d) $x^2y + xy$, e) $x^2y^2/(xy + 1)$. We will consider $x^4 + x^2y^2 + y^4 - 3x - 8y$ as case (f).

We must calculate $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial y})$. The function is homothetic if and only if this ratio is homogeneous of degreee zero in (x, y) by Theorems 20.9 and 20.10.

a) Homothetic: $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial y}) = (2xy + y^2)e^{x^2y}e^{xy^2}/(x^2 + 2xy)e^{x^2y}e^{xy^2} = (2xy + y^2)/(x^2 + 2xy)$, which is H.D.0.

b) Homothetic: $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial y}) = (2/x)/(3/y) = 2y/3x$ which is H.D. 0.

c) Homothetic: $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial y}) = (3x^2y^6 + 6xy^4 + 6y^2)/(6x^3y^5 + 12x^2y^3 + 12xy) = 3y^2/6xy = y/2x$, which is H.D.0.

d) Not Homothetic: $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial y}) = (2xy + y)/(x^2 + x)$, which is not H.D.0. E.g., $f(1,1) = 3/2$ but $f(2,2) = 10/6 = 5/3$.

e) Homothetic: $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial y}) = [(xy + 1)2xy^2 - y(x^2y^2)]/[(xy + 1)2x^2y - x(x^2y^2)] = xy^2(2xy + 2 - xy)/(x^2y)(2xy + 2 - xy) = y/x$, which is H.D.0.

f) Not Homothetic: $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial y}) = (4x^3 + 2xy^2 - 3)/(2x^2y + 4y^3 - 8)$, which is not H.D.0. If it were, $f(0,0) = 3/8 = f(a,a)$ for all $a > 0$ by continuity. But $f(1,1) = -3/2$.