13. Functions of Several Variables

10/01/20

NB: Problems 2, 7, and 25 from Chapter 12, problem 16 from Chapter 13, and problems 4 and 13 from Chapter 29 are due on Tuesday, October 13.

We have previously covered some of the material in Simon and Blume's Chapter 13. We won't cover 13.1 and 13.2, but we will examine continuity (13.4) in rather more detail than they do. We will start with continuity in metric spaces, look at a number of examples, and then consider continuity in general topological spaces. It turns out that the general definition will be useful in metric spaces too.

For functions on the real line, a casual definition is that a function is continuous if you can draw its graph without lifting your pen from the paper. Of course, we will need a more formal definition.

13.1 Continuous Functions in Metric Spaces

We can use convergent sequences to define continuity in metric spaces.

Continuous Functions. Let f map the metric space (X, d_1) into a metric space (Y, d_2). A function f is *continuous at* $\mathbf{x}$ if $f(\mathbf{x}_n) \to f(\mathbf{x})$ whenever $\mathbf{x}_n \to \mathbf{x}$. We say f is *continuous* if it is continuous at every point in X.

One easy continuous function is the identity function defined by $u(\mathbf{x}) = \mathbf{x}$.

► **Example 13.1.1: The Identity Function is Continuous.** We show that u is continuous at any $\mathbf{x} \in X$ by taking any sequence with $\mathbf{x}_n \to \mathbf{x}$. Then $u(\mathbf{x}_n) = \mathbf{x}_n \to \mathbf{x} = u(\mathbf{x})$, showing that u is continuous at $\mathbf{x}$. Since $\mathbf{x}$ was an arbitrary point in X, u is continuous on X. ◄

When the metric space is the normed space ℓ_p^m, each of the coordinate functions $x_i(\mathbf{x}) = x_i$ is continuous.

► **Example 13.1.2: Each Coordinate Function is Continuous in ℓ_p^m.** In ℓ_2^m, we already showed that $\mathbf{x}^n \to \mathbf{x}$ implies each coordinate converges (Theorem 12.7.1).

We will consider the case where $1 \le p < \infty$. The case $p = \infty$ is similar. Let $\mathbf{x}^n \to \mathbf{x}$. Let x_i^n be the i^{th} coordinate of $\mathbf{x}^n$. Then

$$|x_i(\mathbf{x}^n) - x_i(\mathbf{x})| = |x_i^n - x_i| \le \left(\sum_{j=1}^{m} |x_j^n - x_j|^p \right)^{1/p} = \|\mathbf{x}^n - \mathbf{x}\|_p$$

It follows that if $\|\mathbf{x}^n - \mathbf{x}\|_p < \varepsilon$, $|x_i(\mathbf{x}^n) - x_i(\mathbf{x})| < \varepsilon$. Since $\mathbf{x}^n \to \mathbf{x}$, $x_i(\mathbf{x}^n) \to x_i(\mathbf{x})$, showing that x_i is continuous at each $\mathbf{x}$. Because the coordinate function x_i is continuous at every $\mathbf{x} \in \mathbb{R}^m$, it is continuous. ◄

13.2 Examples: Removable and Jump Discontinuities

Not all functions are continuous. The simplest example is a *removable discontinuity*, where we have changed a continuous function at a single point to make it discontinuous. Thus

$$f(x) = \begin{cases} 0 & \text{when } x \neq 2 \\ 50 & \text{when } x = 2 \end{cases}$$

is discontinuous. To see this, observe that

$$\lim_{x \to 2} f(x) = 0 \neq f(2).$$

This type of discontinuity can be removed by changing the value of the function at a single point, in this case, at $x = 2$.

▶ Example 13.2.1: Jump Discontinuity. The function g defined by

$$g(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases}.$$

is not continuous. If $x_n = -1/n$, then $x_n \to 0$, but $\lim_n g(x_n) = 0$, which is not $g(0) = 1$. The function is not continuous at $x = 0$.

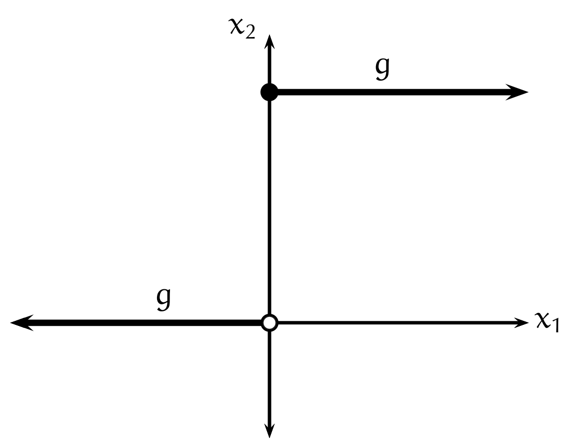

Figure 13.2.2: The function g is zero when $x \leq 0$ and 1 when $x > 0$. This causes a jump discontinuity at $x = 0$.

◀

13.3 Example: More Jump Discontinuities

Functions may have many jump discontinuities. They may even have have infinitely many jumps. We will continue to use the function g defined in Example 13.2.1.

▶ Example 13.3.1: Infinitely Many Discrete Jumps. The function $g(x - x_0)$ has the jump at x_0 instead of 0. The function

$$f(x) = \sum_{k=0}^{\infty} g(x - k)$$

has discontinuities at every non-negative integer. We don't have to worry about using an infinite sum because for any finite x there are only finitely many non-zero terms. In fact, if $x < n$, there are at most n non-zero terms in the sum. ◀

The jumps can also cluster.

▶ Example 13.3.2: Jumps with a Limit. We want to take a sum $\sum_k g(x - x_k)$ where $x_k \to x_0$ where each of the x_k are distinct points. However, we may have problems with convergence. We sidestep this by multiplying the k^{th} term by $1/2^k$. Thus

$$h(x) \sum_{k=1}^{\infty} \frac{1}{2^k} g(x - x_k)$$

has a jump of size $1/2^k$ at x_k. It is continuous at $x_0 = \lim_n x_n$. To show that, let $\varepsilon_k = \min j \le k\{|x_j - x_0|\}$. Choose N so that $|x_n - x_0| < \varepsilon_k$ for $n \ge N$. Then the total jump at the x_n with $n \ge N$ is less than $\sum_{i=k}^{\infty} 2^{-k} < 2^{1-k}$. It follows that $|h(x_n) - h(x_0)| < 2^{1-k}$ for $n \ge N$. Taking the limit shows $|\lim_n h(x_n) - h(x_0)| \le 2^{1-k}$. This holds for every positive integer k, so $\lim_n h(x_n) = h(x_0)$, showing that h is continuous at x_0. ◀

13.4 Example: Singularities

The previous examples involved jump discontinuities. There are other kinds of discontinuities.

▶ Example 13.4.1: Singularity. The function

$$f(x) = \begin{cases} \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

has a discontinuity at zero. This type is called a *singularity*, a term generally applied in real analysis to functions that have an infinite or undefined limit at some finite x_0. The term "singularity" can also be applied when limits of derivatives are ill-behaved. ◀

The following function exhibits another kind of discontinuity.

▶ Example 13.4.2: Essential Singularity.

$$g(x) = \begin{cases} 0 & \text{when } x \leq 0 \\ \sin\left(\frac{1}{x}\right) & \text{when } x > 0. \end{cases}$$

The function g is a type of *topologist's sine curve*. To see that the function is discontinuous, consider the sequence $x_n = 2/n\pi$. Then $g(x_n) = \sin n\pi/2$, which successively takes the values $+1, 0, -1, 0, +1, \ldots$. It simply doesn't converge. It follows that $\lim_{x\to 0} g(x)$ doesn't exist, so g cannot be continuous.

In fact, can find sequences converging to zero where $\lim_n g(x_n)$ takes any value in $[-1, +1]$. This is an example of an *essential singularity*. ◀

13.5 Vector Operations are Continuous

When the metric space we're using is a normed vector space $\big(V, \|\cdot\|\big)$, we can ask about the continuity of the vector operations: vector addition and scalar multiplication. Both of these are continuous. Further, the norm is continuous, and if V is an inner product space, the inner product is also continuous. We will cover each of these in turn over the next several pages.

We'll state the results first, and the proofs will follow over the next four pages. We start with continuity of the norm.

Theorem 13.6.1. *Let $\big(V, \|\cdot\|\big)$ be a normed vector space. Then the norm is a continuous function.*

Vector addition is continuous.

Theorem 13.8.1. *Let $\big(V, \|\cdot\|\big)$ be a normed vector space. Then vector addition is continuous.*

Scalar multiplication is also continuous

Theorem 13.9.1. *Let $\big(V, \|\cdot\|\big)$ be a normed vector space. Then scalar multiplication is continuous.*

If V is an inner product space, the inner product is also continuous.

Theorem 13.10.1. *Let $(V, \cdot)$ be an inner product space. Then $(\mathbf{x}, \mathbf{y}) \to \mathbf{x}\cdot\mathbf{y}$ is continuous.*

A series of remarks after the proofs explains some of the standard tricks and techniques that can help us show convergence and continuity.

13.6 The Norm is Continuous

In normed spaces we can also consider the limit of the norm itself. Not surprisingly, the limit of the norm is the norm of the limit, making the norm continuous.

Theorem 13.6.1. *Let $(V, \|\cdot\|)$ be a normed vector space. Then the norm is a continuous function.*

Proof. We need to show that if $\{x_n\}$ converges to x then $\lim_n \|x_n\| = \|x\|$.
We use the triangle inequality to show

$$\|x\| \leq \|x_n\| + \|x - x_n\| \quad \text{and} \quad \|x_n\| \leq \|x\| + \|x - x_n\|.$$

Together they imply that

$$\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\|$$

Now let $\varepsilon > 0$. We can find N with $\|x_n - x\| < \varepsilon$ for $n \geq N$. It follows that

$$\big|\|x_n\| - \|x\|\big| < \varepsilon$$

for $n \geq N$, so $\|x_n\| \to \|x\|$. Therefore the norm is continuous. □

13.7 Convergent Norms are Bounded

There is a useful corollary that states that the sequence $\{\|x_n\|\}$ is bounded.

Corollary 13.7.1. *Let* $\{x_n\}$ *converge to* x *in a normed vector space* $(V, \|\cdot\|)$. *Then there is a* $B \geq 1$ *with* $\|x_n\| \leq B$.

Proof. Since $\|x_n\| \to \|x\|$, we can set $\varepsilon = 1$ and find an N so that

$$\big|\|x_n\| - \|x\|\big| < 1$$

for $n \geq N$. It follows that

$$\|x_n\| \leq \big|\|x_n\| - \|x\|\big| + \|x\| < 1 + \|x\|$$

for $n \geq N$. Now set

$$B = \max\big\{\|x_1\|, \dots, \|x_{N-1}\|, 1 + \|x\|\big\} \geq 1.$$

The maximum exists because we take the maximum over a finite set. □

13.8 Vector Addition is Continuous

We show the limit of a vector sum is the sum of the limit, proving continuity from $V \times V$ to V.

Theorem 13.8.1. *Let* $(V, \|\cdot\|)$ *be a normed vector space. Then vector addition is continuous.*

Proof. We need to show that if $\{x_n\}$ and $\{y_n\}$ are convergent sequences in V, with limits x and y, that $x_n + y_n \to x + y$

Let $\varepsilon > 0$. Since $x_n \to x$, we can choose N_1 with $\|x_n - x\| < \varepsilon/2$ for $n \geq N_1$. Then choose $N_2 \geq N_1$ with $\|y_n - y\| < \varepsilon/2$ for $n \geq N_2$.

It follows that whenever $n \geq N_2$,

$$\begin{aligned}\|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \\ &\leq \|x_n - x\| + \|y_n - y\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

This shows that $x_n + y_n \to x + y$. □

Triangle Inequality. We rearranged the terms on the top line to group the x and y terms. Then we used the triangle inequality to break the right-hand into two terms on the second line. In the process, we have created two terms that will involve ε. So we use $\varepsilon/2$ as a standard for each to ensure we end up with a single ε in the end.

This is not really necessary. Without it we would end up with 2ε on the right hand side. Since ε is any positive number, 2ε is also any positive number, and works just as well for showing convergence as ε.

What you need to avoid is having things such as n or N in the ultimate right-hand side. These can change with ε, perhaps in an ill-behaved fashion, creating unwanted infinities or indeterminate products such as $0 \times \infty$.

13.9 Scalar Multiplication is Continuous

The other vector operation is scalar multiplication. The theorem says that the limit of the scalar product is the product of the limits.

Theorem 13.9.1. *Let $(V, \|\cdot\|)$ be a normed vector space. Then scalar multiplication is continuous.*

Proof. We need to show that if $\{x_n\}$ a convergent sequence in V with limit x, and $\alpha_n \to \alpha$ in $\mathbb{R}$. Then $\alpha_n x_n \to \alpha x$.

We start by considering the distance between $\alpha_n x_n$ and the proposed limit αx.

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha x_n + \alpha x_n - \alpha x\| \\ &\le \|\alpha_n x_n - \alpha x_n\| + \|\alpha x_n - \alpha x\| \\ &= |\alpha_n - \alpha|\,\|x_n\| + |\alpha|\,\|x_n - x\|\end{aligned}$$

We invoke Corollary 13.7.1 to find a $B \ge 1$ with $\|x_n\| \le B$. Let $\varepsilon > 0$ and choose N_1 so that

$$|\alpha_n - \alpha| < \frac{\varepsilon}{2B}$$

when $n \ge N_1$.

Then choose $N_2 \ge N_1$ with

$$\|x_n - x\| < \frac{\varepsilon}{2(1 + |\alpha|)}$$

for $n \ge N_2$. The one in the denominator avoids any problems that might occur if $\alpha = 0$.

When $n \ge N_2$, both

$$|\alpha_n - \alpha| < \frac{\varepsilon}{2B} \quad \text{and} \quad \|x_n - x\| < \frac{\varepsilon}{2(1 + |\alpha|)}.$$

So for $n \ge N_2$,

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &\le |\alpha_n - \alpha|\|x_n\| + |\alpha|\|x_n - x\| \\ &< \frac{\varepsilon\|x_n\|}{2B} + \frac{\varepsilon|\alpha|}{2(1 + |\alpha|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

That proves the result. □

Adding and Subtracting, Division by Zero. The proof used the old trick of adding and subtracting the same expression so that we can break things up using the triangle inequality, enabling us to deal separately with two simpler terms.

Another problem was the $\|x_n\|$ term, which was bounded using Corollary 13.7.1.

In this case we have a more complicated situation with scaling ε in the other term. If we did not adjust it, we would have ended up with $\varepsilon|\alpha|$ instead of ε. Since $|\alpha|$ is independent of ε, the only potential problem is if we get zero.

In general, when rescaling ε, as we do in the proof, we can't afford to divide by something that might be zero, so we add one before dividing by $|\alpha|$ and by $\|x\|$.

13.10 Inner Products are Continuous

In inner product spaces, the inner product is continuous.

Theorem 13.10.1. *Let* $(V, \cdot)$ *be an inner product space. Then* $(x, y) \to x{\cdot}y$ *is continuous.*

Proof. We need to show that if $\{x_n\}$ and $\{y_n\}$ are convergent sequences with limits $\lim_n x_n = x$ and $\lim_n y_n = y$. Then $\lim_n x{\cdot}_n y_n = x{\cdot}y$.

Let $\varepsilon > 0$ be given. We start by writing

$$\begin{aligned} |x_n{\cdot}y_n - x{\cdot}y| &= |x_n{\cdot}y_n - x{\cdot}y_n + x{\cdot}y_n - x{\cdot}y| \\ &\le |x_n{\cdot}y_n - x{\cdot}y_n| + |x{\cdot}y_n - x{\cdot}y| \\ &\le \|y_n\|\|x_n - x\| + \|x\|\|y_n - y\| \end{aligned}$$

By Corollary 13.7.1, there is a $B \ge 1$ with $\|y_n\| \le B$. Now choose N_1 with $\|x_n - x\| < \varepsilon/2B$ for $n \ge N_1$. Choose $N_2 \ge N_1$ with $\|y_n - y\| < \varepsilon/2\big(1 + \|x\|\big)$ for $n \ge N_2$.

Then for $n \ge N_2$,

$$\begin{aligned} |x_n{\cdot}y_n - x{\cdot}y| &\le \|y_n\|\|x_n - x\| + \|x\|\|y_n - y\| \\ &\le B\frac{\varepsilon}{2B} + \|x\|\frac{\varepsilon}{2\big(1 + \|x\|\big)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

showing that $x_n{\cdot}y_n \to x{\cdot}y$, so the inner product is continuous. □

Bounding the Terms. This proof shows a new technique in addition to some we have seen earlier. There is a troublesome $\|y_n\|$ term in the inequalities. The dependence on n means we can't attempt to make the companion term smaller than $\varepsilon/\big(1 + \|y_n\|\big)$ as the target may converge to zero. This happens if $\|y_n\|$ is unbounded.

We dealt with this by employing Corollary 13.7.1 to bound $\|y_n\|$ from above.

13.11 Continuous Functions, General Case

We currently have three ways to describe a topology: open sets, closed sets, and for metric spaces only, convergent sequences. If we know any one of these we can derive the others. The real point is that any of these can be used to describe continuity.

Before giving the general definition, there is one more piece of notation to introduce. If $f\colon X \to Y$ is a function and the set $B \subset Y$, we define the *inverse image of* B, $f^{-1}(B)$, by $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

Continuous Function (General Definition). A function $f\colon X \to Y$ is *continuous* if and only if $f^{-1}(U)$ is open whenever U is open.

In metric spaces, there are several conditions that are equivalent to continuity.

Theorem 13.11.1. *Suppose* (X, d) *and* (Y, d') *are metric spaces and* $f\colon S \to Y$ *where* $S \subset X$. *The following are equivalent.*

1. f *is continuous in the metric sense. Whenever* $x_n \to x$, $f(x_n) \to f(x)$.
2. $f^{-1}(U)$ *is open whenever* U *is open.*
3. $f^{-1}(A)$ *is closed whenever* A *is closed.*
4. *For all* $\varepsilon > 0$ *and* $x \in S$ *there is a* $\delta > 0$ *such that* $d'\big(f(y), f(x)\big) < \varepsilon$ *whenever* $d(y, x) < \delta$.

Proof. (1) implies (2). **Suppose** $f^{-1}(U)$ **is not open**. Then there is a $x \in f^{-1}(U)$ where every $B_{1/n}(x)$ contains a point not in $f^{-1}(U)$. We can take $x_n \in B_{1/n}(x)$ with $f(x_n) \notin U$ and $x_n \to x$. Now f is continuous, so $f(x_n) \to f(x) \in U$. Since U is open, there is N with $f(x_n) \in U$ for $n \geq N$. But then, $x_n \in f^{-1}(U)$ for $n \geq N$, **contradicting.** $x_n \notin f^{-1}(U)$. Therefore $f^{-1}(U)$ must be open.

(2) if and only if (3). Since $f^{-1}(A^c) = \big[f^{-1}(A)\big]^c$, and A is closed if and only if A^c is open, parts (2) and (3) are equivalent.

(2) implies (4). Now $B_\varepsilon\big(f(x)\big)$ is a open set, so $f^{-1}\big(B_\varepsilon\big(f(x)\big)\big)$ is also open and contains x. It follows that there is a $\delta > 0$ with $B_\delta(x) \subset f^{-1}\big(B_\varepsilon\big(f(x)\big)\big)$, proving (4).

(4) implies (1). Let $\varepsilon > 0$. We can choose $\delta > 0$ so that $f\big(B_\delta(x)\big) \subset B_\varepsilon\big(f(x)\big)$. Now take N so that $x_n \in B_\delta(x)$ whenever $n \geq N$. It follows that $f(x_n) \in B_\varepsilon\big(f(x)\big)$ for $n \geq N$, showing that f is continuous at x. Since $x \in S$ was arbitrary, f is continuous.

The circle (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1) shows that (1), (2), and (4) are equivalent. The fact that (2) $\Leftrightarrow$ (3) means that (3) is equivalent to the other three. □

13.12 ε-δ Continuity

Here's an example using condition (4) to show continuity. It's pretty similar to how we usually show continuity in metric spaces.

▶ **Example 13.12.1: A Quadratic Continuous Function.** Let $f(x) = x^2$. Let $\varepsilon > 0$. Choose $\delta < \min\{1, \varepsilon/(2|x| + 1)\}$. It follows that for $y \in B_\delta(x)$,

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |(x - y)(x + y)| \\ &= |x - y|\,|x + y| \\ &\leq |x - y|\,(2|x| + 1) \\ &\leq \delta(2|x| + 1) \\ &< \varepsilon. \end{aligned} \tag{13.12.1}$$

Because for $|x - y| < \delta$, $|x + y| = |2x - (x - y)| \leq 2|x| + |x - y| \leq 2|x| + 1$. Equation (13.12.1) shows that $f(y) \in B_\varepsilon\big(f(x)\big)$ whenever $y \in B_\delta(x)$, so $f(x) = x^2$ is continuous at any $x \in \mathbb{R}$ by condition (4). ◀

13.13 Inverse Images are Closed

The following example uses condition (3), that the inverse image of closed sets is closed.

▶ **Example 13.13.1: Continuity, Weak Inequalities, and Half-Spaces.** Let $f\colon S \to \mathbb{R}$ be continuous on $S \subset \mathbb{R}^m$. Suppose $\mathbf{x}_n \to \mathbf{x}$ and $f(\mathbf{x}_n) \ge \alpha$. We can rewrite this as $\mathbf{x}_n \in f^{-1}[\alpha, +\infty)$. Since $[\alpha, +\infty)$ is closed, so is its inverse image, and we can conclude that $f(\lim_n \mathbf{x}_n) \ge \alpha$.

It follows that the half-spaces $H^+(\mathbf{p}, \alpha)$ and $H^-(\mathbf{p}, \alpha)$ are both closed sets because $f(\mathbf{x}) = \mathbf{p}\mathbf{x}$ is continuous by Theorem 13.10.1 and $H^+(\mathbf{p}, \alpha) = f^{-1}\big([\alpha, +\infty)\big)$ while $H^-(\mathbf{p}, \alpha) = f^{-1}\big((-\infty, \alpha]\big)$. Both are the inverse images of closed sets.

Moreover, $H(\mathbf{p}, \alpha) = f^{-1}\big(\{\alpha\}\big)$, so the hyperplane $H(\mathbf{p}, \alpha)$ is also closed. Alternatively, we could use the fact that $H(\mathbf{p}, \alpha) = H^+(\mathbf{p}, \alpha) \cap H^-(\mathbf{p}, \alpha)$ to show the hyperplane is closed as the intersection of closed sets. ◀

The budget set is closed for all values of $\mathbf{p}$ and m.

▶ **Example 13.13.2: The Budget Set is Closed.** Recall that the budget set is defined by

$$B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^m_+ : \mathbf{p}{\cdot}\mathbf{x} \le m\}.$$

We can combine the previous example with Example 10.32.1 to see that the budget set is closed. In Example 10.32.1, we found that

$$B(\mathbf{p}, m) = H^-(\mathbf{p}, m) \bigcap \left(\cap_{i=1}^n H^+(\mathbf{e}_i, 0)\right).$$

Then $B(\mathbf{p}, m)$ is closed because it is the intersection of closed half-spaces. ◀

13.14 Combining Continuous Functions

There are a number of ways to make continuous functions from other continuous functions. Most of the standard arithmetic operations: addition, subtraction, multiplication are continuous. We already saw this even in normed spaces for addition, subtraction, and scalar multiplication in Theorems 13.8.1 and 13.9.1.

Another useful way of creating continuous functions from continuous functions is composition.

Theorem 13.14.1. *Suppose* $f: A \to B$ *and* $g: B \to C$ *are both continuous. Then* $g \circ f: A \to C$ *is continuous.*

Proof. Suppose U is open in C. Then $(g \circ f)^{-1}(U) = f^{-1}\big(g^{-1}(U)\big)$. Now $g^{-1}(U)$ is an open subset of B because g is continuous, and so $f^{-1}\big(g^{-1}(U)\big)$ is open because f is continuous. Then $g \circ f$ is continuous. □

Finally, both products and quotients are continuous, as we will show in Theorems 13.15.1 and 13.17.1.

As a consequence, any polynomial

$$p(x) = \sum_{i=0}^{n} a_i x^{n-i} = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

is continuous.

13.15 Products are Continuous

We can use Theorem 13.14.1 in an easy proof that products are continuous

Theorem 13.15.1. *Let* $S \subset \mathbb{R}^m$. *If* $f\colon S \to \mathbb{R}$ *is a continuous real-valued function and* $g\colon S \to \mathbb{R}^m$ *is continuous, then* $f \times g$ *is continuous.*

Proof. We will employ Theorem 13.14.1.

Consider the mapping $F\colon \mathbb{R} \times \mathbb{R}^m$ defined by $F(\alpha, \boldsymbol{x}) = \alpha\boldsymbol{x}$. We know this is continuous by Theorem 13.9.1. Now define the mapping $G\colon \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m$ defined by $G(\boldsymbol{x}) = \big(f(\boldsymbol{x}), g(\boldsymbol{x})\big)$. This is also continuous. It then follows from Theorem 13.14.1 that $F \circ G(\boldsymbol{x}) = f(\boldsymbol{x})g(\boldsymbol{x})$ is continuous. $\square$

It's important in the above proof that $F(\alpha, \boldsymbol{x})$ is jointly continuous, not separately continuous. By *jointly continuous*, we mean that whenever $(\alpha_n, \boldsymbol{x}_n) \to (\alpha, \boldsymbol{x})$, then $F(\alpha_n, \boldsymbol{x}_n) \to F(\alpha, \boldsymbol{x})$ as shown in Theorem 13.9.1. By *separately continuous*, we mean that if $\alpha_n \to \alpha$, then $F(\alpha_n, \boldsymbol{x}) \to F(\alpha, \boldsymbol{x})$ for each $\boldsymbol{x}$ and if $\boldsymbol{x}_n \to \boldsymbol{x}$, then $F(\alpha, \boldsymbol{x}_n) \to F(\alpha, \boldsymbol{x})$ for each α.

13.16 Separately but not Jointly Continuous

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} 0 & \text{when } (x, y) = (0, 0) \\ \dfrac{xy}{x^2 + y^2} & \text{when } (x, y) \neq (0, 0) \end{cases}$$

Suppose $x_n \to 0$. There are two cases.
If $y = 0$, $f(x_n, y) = 0$ for every x_n and so $\lim f(x_n, y) = 0$.
Otherwise, $y \neq 0$. Then

$$f(x_n, y) = \frac{x_n y}{x_n^2 + y^2} \to 0 = f(0, y).$$

Similarly, if $y_n \to 0$, $f(x, y_n) \to 0 = f(x, 0)$. This shows that f is separately continuous in each variable.

But f is not jointly continuous. Let $(x_n, y_n) = (1/n, 1/n)$. Now

$$\begin{aligned} f(x_n, y_n) &= \frac{1}{n^2} \cdot \frac{1}{n^{-2} + n^{-2}} \\ &= \frac{1}{n^2} \cdot \frac{n^2}{2} \\ &= 1/2 \end{aligned}$$

Then $\lim_n f(x_n, y_n) = 1/2 \neq 0 = f(0, 0)$, so f is not jointly continuous at $(0, 0)$. It is jointly continuous everywhere else.

Now $|xy| \leq x^2 + y^2$, so $|f(x, y)| \leq 1/2$. In fact, the full range occurs as a limit along sequences converging to $(0, 0)$. To see this, use polar coordinates and set $(x_n, y_n) = r_n(\cos\theta, \sin\theta)$ with $r_n > 0$ and $r_n \to 0$. Then

$$f(x_n, y_n) = \frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{1}{2}\sin 2\theta.$$

This can take on any value between $-1/2$ and $1/2$.

13.17 Quotients are Continuous

Products and quotients of functions are continuous too, provided they make sense (no division by zero).

Theorem 13.17.1. *The function* $1/x$ *is continuous on* $(0, +\infty)$.

Proof. Suppose $x, x_n > 0$ and $x_n \to x$. Then

$$\left|\frac{1}{x} - \frac{1}{x_n}\right| = \left|\frac{x_n - x}{x x_n}\right| \tag{13.17.2}$$

Choose N_1 so that $|x_n - x| < x/2$ for $n \ge N_1$. Then $-x/2 < x_n - x$, implying $x/2 < x_n$. It follows that $1/x_n < 2/x$ for $n \ge N_1$. Now choose $N_2 \ge N_1$ with $|x_n - x| < \varepsilon x^2/2$. Substituting in equation (13.17.2), we obtain

$$\left|\frac{1}{x} - \frac{1}{x_n}\right| < \frac{\varepsilon x^2}{2}\frac{1}{x x_n} < \frac{\varepsilon}{2x_n} < \varepsilon$$

for $n \ge N_2$. This shows that $x \mapsto 1/x$ is continuous on $(0, +\infty)$. □

Corollary 13.17.2. *If* $g\colon X \to (0, +\infty)$ *with* $X \subset \mathbb{R}^k$*, then* $f(\mathbf{x}) = 1/g(\mathbf{x})$ *is continuous on* X.

Proof. This follows from Theorems 13.17.1 and 13.14.1. □

A New Challenge. We have a new challenge here connected with the fact that a $1/|x_n|$ term appears in the inequalities. This can blow up if $|x_n| \to 0$. The way to deal with is to show that $|x_n|$ must stay away from 0. In fact, we show it is eventually bounded below by the non-zero number $x/2$, so $1/|x_n|$ is bounded above by $2/x$.

29. Limits and Compact Sets

This section covers various topics from Chapter 29, including boundedness, monotone sequences, Dedekind completeness, completeness, and compactness. Section 29.3, on connected sets, will be covered later.

29.1 Bounded Sets

For this section we restrict our attention to normed vector spaces $(V, \|\cdot\|)$.

Bounded. Let $(V, \|\cdot\|)$ be a normed vector space. A set $S \subset V$ is *bounded* if there is some number $K > 0$ with $\|\mathbf{x}\| \le K$ for all $\mathbf{x} \in S$.

Alternatively, a set is bounded if it is contained in some ball about zero, $B_K(\mathbf{0})$. In any ℓ_p^m, $|x_i| \le \|\mathbf{x}\|_p$, so $\|\mathbf{x}\|_p \le K$ implies $|x_i| \le K$ for $i = 1, \dots, m$.

► **Example 29.1.1: Bounded Budget Sets.** The budget set $B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^m_+ : \mathbf{p}\cdot\mathbf{x} \le m\}$ is bounded in any ℓ_p^m when $\mathbf{p} \gg \mathbf{0}$, but is not bounded if some $p_i = 0$.

When $\mathbf{p} \gg \mathbf{0}$ and $\mathbf{x} \in B(\mathbf{p}, m)$, $p_i x_i \le m$, so $0 \le x_i \le m/p_i$. Let $K_0 = \max_i\{m/p_i\}$.

When $p = \infty$, $\|\mathbf{x}\|_\infty \le K_0$, so the budget set is ℓ_∞^m bounded.

When $1 \le p < \infty$, $\sum_{i=1}^m |x_i|^p \le mK_0^p$, implying $\|\mathbf{x}\|_p \le m^{1/p}K_0$. Set $K_p = m^{1/p}K_0$ Then $\|\mathbf{x}\|_p \le K_p$, whenever $\mathbf{x} \in B(\mathbf{p}, m)$. This shows that the budget set is bounded in ℓ_p^m whenever $\mathbf{p} \gg \mathbf{0}$. ◄

► **Example 29.1.2: Unbounded Budget Set.** If some $p_i = 0$, we can increase x_i without bound and still stay in the budget set. So the budget set is not bounded. For example, in $\mathbb{R}^2$, set $\mathbf{p} = (1, 0)$ and $m = 1$. Then $B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^2_+ : 0 \le x_1 \le 1\}$ is unbounded, as illustrated in Figure 29.1.3.

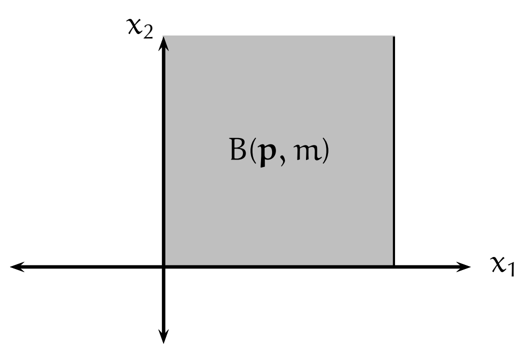

Figure 29.1.3: Here the price vector is $\mathbf{p} = (1, 0)$, resulting in an unbounded budget set.

◄

29.2 Boundedness in Metric Spaces

The absolute homogeneity of degree one means that norms measure uniformly throughout their vector space. That may not happen with metrics, where the analog of boundedness doesn't bound.

► Example 29.2.1: Boundedness Unbounded in a Metric Space. Consider $\mathbb{R}^2$ with the metric used in Figure 10.28.1:

$$d(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\frac{|x_1 - y_1|}{1 + |x_1 - y_1|} + \frac{1}{4}\frac{|x_2 - y_2|}{1 + |x_2 - y_2|}$$

Suppose we have a set S where $d(\mathbf{0}, \mathbf{x}) \leq 1/4$ for all $\mathbf{x} \in S$. In spite of the bound on the metric, this set is not bounded!

In fact, $d\big(\mathbf{0}, (0, y_2)\big) = |y_2|/4\big(1 + |y_2|\big) < 1/4$, so the entire vertical axis is part of S. See also Figure 29.2.2 below.[1]

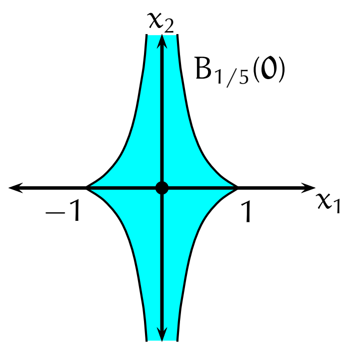

Figure 29.2.2: Although the sequence metric is bounded, that cannot be said about the ball of radius $1/4$ about $\mathbf{0}$. We restrict the metric to $\mathbb{R}^2$ in the diagram. The cyan area illustrates points $\mathbf{x}$ with $d(\mathbf{x}, \mathbf{0}) = 1/4$ in $\mathbb{R}^2$. The entire vertical axis has $d(\mathbf{x}, \mathbf{0}) < 1/4$, with $d\big((0, x_2), 0\big) \to 1/4$ as $x_2 \to \pm\infty$.

◄

[1] You may recognize this set from Figure 10.28.1.

29.3 Upper Bounds in $\mathbb{R}$

In the real line, a set is bounded if and only if there are A and B with $A \leq x \leq B$ for every $x \in S$. In other words, we have two bounds, a lower bound A and upper bound B.

The most important of all upper bounds is the smallest of them, the *least upper bound* or *supremum*.

Upper Bounds. Let $S \subset \mathbb{R}$. An *upper bound for* S is a number B with $x \leq B$ for all $x \in S$. The *least upper bound* or *supremum of* S is the smallest number C with $x \leq C$ for all $x \in S$. That is, if B is an upper bound for S, then $B \geq C$. We denote the least upper bound (supremum) of S by $\sup S$.

If $I = (a, b)$, $\sup I = b$ as b is an upper bound for the interval I and no smaller number is an upper bound for I.

If $S = \{x \in \mathbb{Q} : x^2 < 2\}$, any rational number greater that $\sqrt{2}$ is a upper bound for S and no rational number smaller than $\sqrt{2}$ is an upper bound for S. To see the latter, if $y < \sqrt{2}$, write enough digits of $\sqrt{2}$ to get a rational number larger than y which is still in S. Then $\sup S = \sqrt{2}$.

Let $S = \{x \in \mathbb{Q} :$ there is a circle with diameter d, circumference c and $c/d > x\}$. Here $\sup S = \pi$.

The following axiom (or one that performs a similar role) is part of the definition of the real numbers.

Dedekind Completeness Axiom. *Every non-empty set S of real numbers that is bounded above has a supremum* $\sup S$. *Moreover,* $\sup S$ *is a real number.*

29.4 Lower Bounds in $\mathbb{R}$

We can also consider lower bounds. Here too, one is most important, the *greatest lower bound* or *infimum*.

Lower Bounds. Let $S \subset \mathbb{R}$. A *lower bound for* S is a number D with $x \geq D$ for all $x \in S$. The *greatest lower bound* or *infimum of* S is the smallest number C with $x \geq C$ for all $x \in S$. That is, if D is an lower bound for S, then $D \leq C$. We denote the greatest lower bound (infimum) of S by $\inf S$.

It's now easy to prove that sets in $\mathbb{R}$ have infima by using Dedekind completeness.

Theorem 29.4.1. *Every non-empty set* S *of real numbers that is bounded below has a infimum,* $\inf S$*. Moreover,* $\inf S$ *is a real number.*

Proof. Consider $-S = \{-x : x \in S\}$. It is bounded above and so has a supremum $\sup -S$ by Dedekind completeness. Then $\inf S = -(\sup -S)$. □

In $\mathbb{R}$ a set is *bounded* if and only if it has both upper and lower bounds.

If a non-empty set S is not bounded below, we set $\inf S = -\infty$. Similarly, a set that is not bounded above has $\sup S = +\infty$.

What about the empty set? If S is empty, $\inf \emptyset = +\infty$. and $\sup \emptyset = -\infty$. The rationale is that every number is a lower/upper bound for the empty set. This is vacuously true in that for every $a \in \mathbb{R}$, there is nothing in the empty set that is smaller or larger. Hence a is both an upper and lower bound. With all real numbers being both upper and lower bounds for the empty set, it greatest lower bound is $\inf \emptyset = \sup \mathbb{R} = +\infty$, while the least upper bound is $\sup \emptyset = \inf \mathbb{R} - \infty$.

By allowing the values $\pm\infty$, every set of real numbers has an infimum and a supremum.

29.5 Monotone Sequences in $\mathbb{R}$

Monotone Sequences. A sequence $\{x_n\}$ of real numbers is *monotone increasing* if $x_n \le x_{n+1}$ for $n = 1, 2, \ldots$. It is *monotone decreasing* if $x_n \ge x_{n+1}$ for $n = 1, 2, \ldots$. Saying a sequence is *monotone* means it is either monotone increasing or monotone decreasing.

Thus $x_n = 1/n$ is monotone decreasing and $x_n = n^2$ is monotone increasing.

Theorem 29.5.1. *Every bounded monotone sequence in $\mathbb{R}$ converges. Moreover, if x_n is increasing, $\lim_n x_n = \sup_n x_n$ and if x_n is decreasing, $\lim_n x_n = \inf_n x_n$.*

Proof. We will prove the decreasing case since Simon and Blume (Theorem 29.2) do the increasing case.

Let $x = \inf_n x_n$. I claim $x_n \to x$. Let $\varepsilon > 0$. Then $x + \varepsilon$ is not a lower bound for $\{x_n\}$, so there is N with $x \le x_N < x + \varepsilon$. Because x_n is monotone decreasing and x is a lower bound for the sequence, $x \le x_n \le x_N < x + \varepsilon$ for $n \ge N$. But then $|x_n - x| < \varepsilon$ for $n \ge N$ which shows that $\lim_n x_n = x$. □

The notations $x_n \downarrow x$ and $x_n \uparrow x$ are sometimes used to indicate monotone convergence to x.

We can extract a monotone subsequence from every sequence of real numbers.

Theorem 29.5.2. *Every sequence of real numbers has a monotone subsequence.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. **If** it has a monotone increasing subsequence, we are done.

Else, the sequence has no monotone increasing subsequence. Take x_i in the sequence. Then choose a $j > i$ with $x_j \ge x_i$, then a k with $k > j$ and $x_k \ge x_j$. Since there are no monotone increasing subsequences, this process ends after a finite number of steps (possibly one).

Call the highest number in the sequence a *dominant element*. Suppose it is x_ℓ. Then $x_\ell > x_n$ for all $n > \ell$. Then find next the dominant element following x_ℓ and repeat. This gives us a monotone decreasing subsequence of successive dominant elements and we are done. □

29.6 Cauchy Sequences

10/06/20

We still lack a criterion to tell if a sequence converges or not. At present, all we can do is try to find a limit. There is a test for convergence. Whether a sequence in $\mathbb{R}$ converges depends on whether it is a Cauchy sequence.

Cauchy Sequence. A sequence $\{x_n\}$ in a metric space (X, d) is a *Cauchy sequence* if for every $\varepsilon > 0$ there is an N with $d(x_n, x_m) < \varepsilon$ whenever $m, n \geq N$.

In other words, a sequence is a Cauchy sequence if the terms of the sequence get closer together.

Theorem 29.6.1. *Suppose $\{x_n\}$ is a sequence in a metric space (X, d). If $x_n \to x$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$ be given. Since $x_n \to x$, there is an N with $d(x_n, x) < \varepsilon/2$ for $n \geq N$. Then for $m, n \geq N$, $d(x_m, x) < \varepsilon/2$ and $d(x_n, x) < \varepsilon/2$. It follows that $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \varepsilon$ for $n, m \geq N$, showing that $\{x_n\}$ is a Cauchy sequence. □

But is there a converse? Do Cauchy sequences converge? This is more difficult to prove. We focus our attention on $\mathbb{R}$ and start by showing Cauchy sequences in $\mathbb{R}$ are bounded.

Theorem 29.6.2. *Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}$ is bounded.*

Proof. Set $\varepsilon = 1$ and choose N such that $|x_n - x_m| < 1$ for $n, m \geq N$. Then $|x_n - x_N| < 1$ for $n \geq N$. Now $B = \max\{|x_n| : n \leq N\} + 1$ is a upper bound for $\{|x_n|\}$ since $|x_n| \leq |x_N| + |x_n - x_N| < |x_N| + 1 \leq B$ for $n \geq N$, and $|x_n| \leq B$ for $n \leq N$ by definition. □

29.7 Real Cauchy Sequences are Convergent

We can now use our previous theorems to show that every Cauchy sequence of real numbers converges.

Theorem 29.7.1. *Let* $\{x_n\}$ *be a Cauchy sequence in* $\mathbb{R}$. *Then* $\{x_n\}$ *converges.*

Proof. By Theorem 29.5.2, x_n has a monotone subsequence x_{n_k}. By Theorem 29.6.2, the subsequence is bounded, so by Theorem 29.5.1, the subsequence x_{n_k} converges to some $x \in \mathbb{R}$.

Let $\varepsilon > 0$. Choose N such that $|x_n - x_m| < \varepsilon/2$ whenever $m, n \geq N$. Now find $n_k \geq N$ with $|x_{n_k} - x| < \varepsilon/2$. It follows that for all $m \geq N$,

$$|x - x_m| \leq |x - x_{n_k}| + |x_{n_k} - x_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

showing that $x_n \to x$. $\square$

29.8 Complete Metric Spaces

A metric space is *complete* if every Cauchy sequence converges. Theorem 29.7.1 showed that $(\mathbb{R}, |\cdot|)$ is a complete normed space, hence a complete metric space.

It's not hard to show that $\mathbb{R}^m$ is complete in the Euclidean norm.

Theorem 29.8.1. *The normed space ℓ_2^m is complete.*

Proof. Let $\varepsilon > 0$. Choose N such that for $k, n \geq N$, $\|\mathbf{x}^k - \mathbf{x}^n\|_2 < \varepsilon$. Let x_i^k be the i^{th}component of $\mathbf{x}^k$. Then

$$|x_i^k - x_i^n| \leq \|\mathbf{x}^k - \mathbf{x}^n\|_2 < \varepsilon$$

for all $k, n \geq N$. It follows that each $\{x_i^n\}$ is a Cauchy sequence.

Then each $x_i^n \to x^i$ for some $x_i \in \mathbb{R}$. By Theorem 12.7.1, $\mathbf{x}^n \to \mathbf{x}$, showing that $(\mathbb{R}^m, \|\cdot\|_2)$ is complete. □

The same sequences converge in all of the ℓ_p norms, so each ℓ_p^m is complete.

Special Spaces. A *Banach space* is a normed space that is complete in the metric defined by the norm. It follows that any ℓ_p^m is a Banach space. As a matter of fact, so are ℓ_p and L^p.

A *Hilbert space* is an inner product space that is complete under the norm generated by the inner product. Thus ℓ_2^m is a Hilbert space, as are ℓ_2 and L^2.

Finally, a vector space that is a complete metric space, with continuous linear operations, is called a *Fréchet space*. The sequence space $\mathbf{s}$ is a Fréchet space.

29.9 Spaces of Continuous Functions

Another normed space that is complete is the space of continuous bounded functions on a set X with values in $\mathbb{R}^m$ and convergence defined by the supremum norm.

The Normed Space $\left(\mathcal{C}_b(X;\mathbb{R}^m), \|\cdot\|_\infty\right)$. Let X be a subset $\mathbb{R}^m$. The space of continuous bounded functions on X with values in $\mathbb{R}^m$ is $\mathcal{C}_b(X;\mathbb{R}^m) = \{\mathbf{f}\colon X \to \mathbb{R}^m : \mathbf{f}$ is continuous and bounded$\}$. It has the supremum norm

$$\|\mathbf{f}\|_\infty = \sup\left\{|f_i(\mathbf{x})| : \mathbf{x} \in X, i = 1, \ldots, m\right\}$$

where f_i is the i^{th} component of $\mathbf{f}$. When $m = 1$, we may write $\mathcal{C}_b(X)$ instead of $\mathcal{C}_b(X;\mathbb{R}^m)$.

By saying $\mathbf{f}$ is bounded over X, we mean there is a $B > 0$ with $|f_i(\mathbf{x})| \leq B$ for every $i = 1, \ldots, m$ and $\mathbf{x} \in X$. By the Dedekind Completeness Axiom, $\|\mathbf{f}\|_\infty$ exists. Moreover, $\|\mathbf{f}\|_\infty \leq B$. Dedekind completeness allows us to define the norm as we did.

29.10 Pointwise Convergence

There are two types of convergence in $\mathcal{C}_b(X;\mathbb{R}^m)$. The first is pointwise convergence.

Pointwise Convergence. A sequence of functions $\mathbf{f}_n : X \to \mathbb{R}^m$ *converges pointwise to* $\mathbf{f}$ if $\lim_n \mathbf{f}_n(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in X$.

Unfortunately, the pointwise limit of continuous functions need not be continuous. The space of bounded continuous functions $\mathcal{C}_b(X;\mathbb{R}^m)$ is not pointwise complete.

► Example 29.10.1: Discontinuous Limit of Continuous Functions. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} nx & \text{for } x \in [0, 1/n)] \\ 1 & \text{for } x \in [1/n, 1]. \end{cases}$$

The f_n are continuous functions on $[0,1]$ because both definitions match at $x = 1/n$.

The pointwise limit is

$$f(x) = \begin{cases} 0 & \text{when } x = 0 \\ 1 & \text{for } x \in (0, 1]. \end{cases}$$

Of course, the limit function f is discontinuous at 0. It is illustrated in Figure 29.10.2. ◄

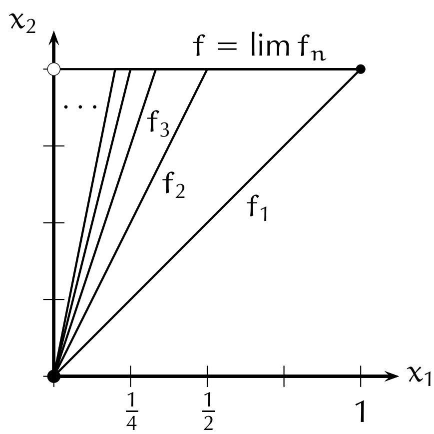

Figure 29.10.2: The functions f_n described in Example 29.10.1 converge upwards to f, which is the line at the top, except at zero, where it is zero, giving us a pointwise limit of continuous functions that is not continuous at zero.

One of the limitations of Riemann integration is that pointwise limits of Riemann integrable functions need not be Riemann integrable.

29.11 Uniform Convergence in $\mathcal{C}_b(X;\mathbb{R}^m)$

The second type of convergence is uniform convergence, convergence in the supremum norm.

Uniform Convergence. A sequence of functions $\mathbf{f}_n \colon X \to \mathbb{R}^m$ *converges uniformly to* $\mathbf{f}$ if $\|\mathbf{f}_n - \mathbf{f}\|_\infty \to 0$.

Unlike pointwise limits, uniform limits of functions in $\mathcal{C}_b(X;\mathbb{R}^m)$ are bounded continuous functions.

We will need to distinguish two supremum norms, on $\mathbb{R}^m$ and on $\mathcal{C}_b(X;\mathbb{R}^m)$. We denote the supremum norm on $\mathbb{R}^m$ by $\|\mathbf{x}\|_{m,\infty} = \sup_i |x_i|$ and the supremum norm on $\mathcal{C}_b(X;\mathbb{R}^m)$ by $\|\mathbf{f}\|_\infty$. They are related by the equation

$$\|\mathbf{f}\|_\infty = \sup\left\{\|\mathbf{f}(\mathbf{x})\|_{m,\infty} : \mathbf{x} \in X\right\}$$

so

$$\|\mathbf{f}(\mathbf{x})\|_{m,\infty} \le \|\mathbf{f}\|_\infty$$

for all $\mathbf{x} \in X$.

29.12 Uniform Limits of Continuous Functions are Continuous

One important result is that the uniform limit of continuous functions is continuous, something that cannot always be said about the pointwise limit.

In the theorem below, we use $\mathbf{f}^n$ to denote a sequence of functions in $\mathbb{R}^m$ so that we can use f_i^n for its i^{th} component.

Theorem 29.12.1. *Suppose* $\mathbf{f}^n \in \mathcal{C}_b(X;\mathbb{R}^m)$ *converges uniformly to a limit* $\mathbf{f}$*. Then* $\mathbf{f} \in \mathcal{C}_b(X;\mathbb{R}^m)$.

Proof. We first show that $\mathbf{f}$ is a bounded function. This follows from the triangle inequality. Choose N with $\|\mathbf{f}^n - \mathbf{f}\|_\infty < 1$ for $n \ge N$. Then

$$\|\mathbf{f}\|_\infty \le \|\mathbf{f} - \mathbf{f}^N\|_\infty + \|\mathbf{f}^N\|_\infty \le 1 + \|\mathbf{f}^N\|_\infty.$$

It follows that each f_i is also bounded because

$$|f_i(\mathbf{x})| \le 1 + \|\mathbf{f}^N(\mathbf{x})\|_{m,\infty} \le 1 + \|\mathbf{f}^N\|_\infty$$

for all $\mathbf{x} \in X$ and $i = 1, \dots, m$.

Next we show that each f_i is continuous. Suppose $\mathbf{x}_k \to \mathbf{x}$ and let $\varepsilon > 0$ be given. Choose N with $\|\mathbf{f}^n - \mathbf{f}\|_\infty < \varepsilon$ for $n \ge N$. By continuity of $\mathbf{f}^N$, we may then choose $K \ge N$ with $\|\mathbf{f}^N(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x})\|_{m,\infty} < \varepsilon$ for $k \ge K$. We then have

$$\begin{aligned}
&|f_i(\mathbf{x}_k) - f_i(\mathbf{x})| \\
&\qquad \le \|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}(\mathbf{x})\|_{m,\infty} \\
&\qquad \le \|\mathbf{f}(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x}_k)\|_{m,\infty} + \|\mathbf{f}^N(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x})\|_{m,\infty} + \|\mathbf{f}^N(\mathbf{x}) - \mathbf{f}(\mathbf{x})\|_{m,\infty} \\
&\qquad \le \|\mathbf{f} - \mathbf{f}^N\|_\infty + \|\mathbf{f}^N(\mathbf{x}_k) - \mathbf{f}^N(\mathbf{x})\|_{m,\infty} + \|\mathbf{f} - \mathbf{f}^N\|_\infty \\
&\qquad < 3\varepsilon
\end{aligned}$$

when $k \ge K$. Since ε was any positive number, this shows that each f_i and so $\mathbf{f}$ is continuous at $\mathbf{x}$. Since $\mathbf{x}$ was any point in X, $\mathbf{f}$ is continuous on all of X. □

Three-ε Argument. This is our first example of a 3-ε argument. One of its uses is to show continuity of the limit. Here the right-hand side is broken into three parts by twice adding and subtracting, and then using the triangle inequality.

In this case, two of the parts are made smaller than ε by choosing N large, the third part is made small by using continuity of a particular $\mathbf{f}^N$ and making sure $\mathbf{x}_k$ and $\mathbf{x}$ are close enough together.

29.13 $\mathcal{C}_b(X;\mathbb{R}^m)$ is Uniformly Complete

The important fact about the uniform topology on $\mathcal{C}_b(X;\mathbb{R}^m)$ is that the uniform limit of continuous functions is continuous and that every uniformly Cauchy sequence has a uniform limit. Together, they show that $\mathcal{C}_b(X;\mathbb{R}^m)$ is a complete metric space in the uniform topology.

Theorem 29.13.1. *The space $\big(\mathcal{C}_b(X;\mathbb{R}^m), \|\cdot\|_\infty\big)$ is a complete metric space, and so a Banach space.*

Proof. Suppose $\{\mathbf{f}^n\}$ is a Cauchy sequence in $\mathcal{C}_b(X;\mathbb{R}^m)$.

Step one is to find the limit. Let $\varepsilon > 0$. Then there is an N with $\|\mathbf{f}^k - \mathbf{f}^n\|_\infty < \varepsilon$ for $k, n \ge N$. It follows that for every $\mathbf{x} \in X$,

$$|f_i^k(\mathbf{x}) - f_i^n(\mathbf{x})| \le \|\mathbf{f}^k - \mathbf{f}^n\|_{m,\infty} \le \|\mathbf{f}^k - \mathbf{f}^n\|_\infty < \varepsilon$$

for all $k, n \ge N$ and $i = 1, \ldots, m$. In other words, each $\{f_i^n(\mathbf{x})\}$ is a Cauchy sequence and so has a limit $f_i(\mathbf{x})$. At this point we don't know whether $\mathbf{f}^n$ converges uniformly to $\mathbf{f}$. We only know that it converges pointwise to $\mathbf{f}$.

We finish the proof off by showing $\mathbf{f}^n$ converges uniformly to $\mathbf{f}$. At that point Theorem 29.12.1 shows that the limit is in $\mathcal{C}_b(X;\mathbb{R}^m)$.

Choose a new $\varepsilon > 0$ and N with $\|\mathbf{f}^k - \mathbf{f}^n\|_\infty < \varepsilon/2$ for $k, n \ge N$. Then for $\mathbf{x} \in X$,

$$|f_i^k(\mathbf{x}) - f_i^n(\mathbf{x})| \le \|\mathbf{f}^k - \mathbf{f}^n\|_\infty < \varepsilon/2.$$

Taking the limit as $k \to \infty$ shows that $|f_i(\mathbf{x}) - f_i^n(\mathbf{x})| \le \varepsilon/2$ for every $\mathbf{x} \in X$ and $i = 1, \ldots, m$. Then we can take the supremum over all $i = 1, \ldots, m$, and $\mathbf{x} \in X$ to find $\|\mathbf{f} - \mathbf{f}^n\|_\infty \le \varepsilon/2 < \varepsilon$ for $n \ge N$. In other words, $\mathbf{f}^n \to \mathbf{f}$ uniformly, not just pointwise. □

You'll notice that the functions in Example 29.10.1 do not converge uniformly. In fact, consideration of $x = 0$ shows that $\|f - f_n\|_\infty = 1$ for all n.

29.14 Compact Sets: Introduction

One type of set of particular importance are compact sets. We will often be able to show that functions defined on compact sets have maxima and/or minima. This will insure that standard economic problems such as utility maximization, expenditure minimization, cost minimization and profit maximization have solutions.

One of the odd things about compact sets is that we have not one, not two, but three definitions. When they all make sense, they are equivalent, but they don't all always apply. We start with a preliminary definition.

Cover. Let $S \subset X$. Any collection $\mathcal{U}$ of subsets of X obeying

$$S \subset \bigcup_{U \in \mathcal{U}} U$$

is called a *cover* of S. If $\mathcal{U}$ consists solely of open sets, we call $\mathcal{U}$ an *open cover* and if the collection $\mathcal{U}$ is finite, it is a *finite cover*.

If $\mathcal{U}$ is a cover of S, a *subcover* of $\mathcal{U}$ is a collection $\mathcal{V} \subset \mathcal{U}$ that also covers S.

29.15 Compact Sets: Definitions

There are three types of compactness. The first type applies to $\mathbb{R}^m$ with the usual topology. The second works in any metric space. The third applies to every topological space. Fortunately, whenever two or more apply, they all agree on which sets are compact.

Compact Set.

1. A set $S \subset \mathbb{R}^m$ is *closed and bounded compact* if it is closed and bounded.
2. A set S in a metric space (X, d) is *sequentially compact* if every sequence in S has a subsequence that converges to an element of S.
3. A set S is *Heine-Borel compact* if whenever $\{U_\alpha\}$ are open sets covering S, there is a finite subcover of S, $\{U_{\alpha_i}\}_{i=1}^N$.

These definitions are equivalent when two or more apply.

Consider $S = [a, b] \subset \mathbb{R}$. The set S is compact. We know that the closed interval is closed, and it is bounded with bound $K = \max\{|a|, |b|\}$. Closed balls in $\mathbb{R}^m$ are also closed and bounded, hence compact.

Half-open intervals such as $(a, b]$ are bounded, but not closed (e.g., a is a limit point of the set, but not part of the set).

Theorem 29.15.1. *Suppose $S \subset \mathbb{R}^m$ is sequentially compact in the usual topology. Then it is closed and bounded.*

Proof. Let $\{x_n\}$ be a sequence in S with $x_n \to x$. Since S is sequentially compact, $x \in S$, showing that S is closed.

Now suppose S is not bounded. Then for every integer $n > 0$ there is a $x_n \in S$ with $\|x_n\| > n$. By sequential compactness this has a convergent subsequence x_{n_k} with $x_{n_k} \to x \in S$. By Theorem 13.6.1, $\|x_{n_k}\| \to \|x\| < \infty$. But by construction, $\|x_{n_k}\| \to \infty$. This **contradiction shows** that S **must be bounded**. $\square$

29.16 The Bolzano-Weierstrass Theorem

Because we have done things in a different order than Simon and Blume, we can use a different proof for the Bolzano-Weierstrass Theorem.

Bolzano-Weierstrass Theorem. *Any box* $B = \prod_{i=1}^{m}[a_i, b_i] \subset \mathbb{R}^m$ *is sequentially compact.*

Proof. Let $\{\mathbf{x}^n\}$ be a sequence in B and let x_i^n be the i^{th} coordinate of $\mathbf{x}^n$. By Theorem 29.5.2, x_1^n has a monotone subsequence $x_1^{n_k^1}$. Similarly, $x_2^{n_k^1}$ has a monotone subsequence $x_2^{n_k^2}$. Continue taking subsequence until we run out of coordinates, obtaining $\mathbf{y}^k = \mathbf{x}^{n_k^m}$ which is monotone in every coordinate. It is also bounded in every coordinate, and so converges in every coordinate by Theorem 29.5.1. Finally, Theorem 12.7.1 shows this subsequence converges in the ℓ_2 norm. Since B is closed, $\lim_k \mathbf{y}^k \in B$, showing that B is sequentially compact. □

Lemma 29.16.1. *If S is sequentially compact and T is a closed subset of S, then T is sequentially compact.*

Proof. Let $\{\mathbf{x}_n\}$ be a sequence in $T \subset S$. Since S is sequentially compact, we can find a subsequence $\mathbf{x}_{n_k}$ that converges to a point in S. As T is closed, and each $\mathbf{x}_{n_k} \in T$, $\lim_k \mathbf{x}_{n_k} \in T$, showing that T is sequentially compact. □

We can now show that closed and bounded sets in $\mathbb{R}^m$ are sequentially compact. Since we already showed in Theorem 29.15.1 that sequentially compact sets in $\mathbb{R}^m$ are closed and bounded, the two definitions of compactness are equivalent on $\mathbb{R}^m$.

Corollary 29.16.2. *A set S is a closed and bounded set in* $\mathbb{R}^m$*, if and only if it is sequentially compact.*

Proof. Only If: Since S is bounded, it is contained in $[-K, K]^m$ for some K. Now $[-K, K]^m$ is sequentially compact by the Bolzano-Weierstrass Theorem, and S is a closed subset of $[-K, K]^m$, so S is also sequentially compact.

The converse was shown in Theorem 29.15.1. □

29.17 Heine-Borel Compact

It's easy to show that a closed subset of a Heine-Borel compact set is Heine-Borel compact.

Theorem 29.17.1. *Suppose S is Heine-Borel compact and $T \subset S$ is closed. Then T is Heine-Borel compact*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of T. Since T^c is open, $\{T^c\} \cup \{U_\alpha\}_{\alpha \in A}$ is an open cover of S. It has a finite subcover, $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ and possibly T^c. It follows that $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ is a finite subcover of T (T^c can never cover any part of T). Thus T is Heine-Borel compact. □

Theorem 29.17.2. *A set S in a metric space (X, d) is Heine-Borel compact if and only if it is sequentially compact.*

Proof. Only if case: Here S is Heine-Borel compact. **If S is not sequentially compact**, there must be a sequence $\{x_n\}$ that has no convergent subsequence. It follows that no point in $\{x_n\}$ can be repeated infinitely often. Otherwise we could take a subsequence that is only that point, and so converges.

For any $x \in S$, if for every $\varepsilon > 0$, $B_\varepsilon(x)$ contains one of the x_n, we can construct a convergent subsequence by taking $\varepsilon = 1/k$ and choosing $x_{n_k} \in B_{1/n}(x)$. Then $\lim_k x_{n_k} = x$. As this is impossible, there is always an ε_x with $B_{\varepsilon_x}(x)$ containing no x_n except possibly x.

The $B_{\varepsilon_x}(x)$ are an open cover of S, so they have a finite subcover. Now the union of finitely many of these balls contains at most finitely many terms of the sequence $\{x_n\}$—it's important here that no point is infinitely repeated. It follows that the open subcover doesn't cover S! **This contradiction** implies that S is sequentially compact, that every sequence has a convergent subsequence.

If case: Omitted, it takes us too far afield. □

29.18 Continuous Maps and Compactness

The continuous image of a compact set is compact.

Theorem 29.18.1. *Let* $f: X \to Y$ *where* X *and* Y *are topological spaces. If* f *is continuous and* $S \subset X$ *is compact,* $f(S)$ *is also compact.*

Proof. Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of $f(S)$. Let $U_\alpha = f^{-1}(V_\alpha)$. Then the colloection $\{U_\alpha\}$ is an open cover of A. It has a finite subcover $U_{\alpha_1}, \ldots, U_{\alpha_k}$. It follows that $V_{\alpha_1}, \ldots, V_{\alpha_k}$ is a finite subcover of $f(S)$. Thus $f(S)$ is compact. □

29.19 Weierstrass's Theorem

For our purposes, one of the most useful result from topology is Weierstrass's Theorem. It is the key to showing that many economic problems, such as the consumer's utility maximization problem, have solutions.[2]

Weierstrass's Theorem. *Let $S \subset \mathbb{R}^m$ be compact and $f\colon S \to \mathbb{R}$ be continuous. Then there are $\mathbf{x}_* \in S$ and $\mathbf{x}^* \in S$ with $f(\mathbf{x}_*) \le f(\mathbf{x}) \le f(\mathbf{x}^*)$ for all $\mathbf{x} \in S$. In other words, f attains both a maximum and minimum on S.*

Proof. Now f is continuous and S compact, so $f(S)$ is compact by Theorem 29.18.1. Since $f(S) \subset \mathbb{R}$, it is closed and bounded. Thus $\sup f(S) \in f(S)$ and $\inf f(S) \in f(S)$. Taking $x^* \in S$ with $f(x^*) = \sup f(S)$ and $x_* \in S$ with $f(x_*) = \inf f(S)$ completes the proof. □

► **Example 29.19.1: Cases where Weierstrass Doesn't Apply.** If the set is not compact, a maximum may not exist. For example, if $S = [0, 1)$ and $f(x) = x$, there is no maximum. It would be at 1, if 1 were in S.

If the function is not continuous, a maximum may not exist. Let $S = [0, 1]$ and define $f(x) = x$ for $x < 1$ and $f(1) = 0$. Once again, the problem occurs at $x = 1$, but here the problem is that the function jumps downward. ◄

[2] Weierstrass's Theorem is not in Chapter 29 of Simon and Blume, but it is convenient to cover it now. We have borrowed Weierstrass's Theorem from Chapter 30.1

29.20 Utility Maximization

We consider the problem of maximizing a continuous utility function $u\colon \mathbb{R}^m \to \mathbb{R}$ under the budget constraint $\mathbf{p}\cdot\mathbf{x} \le m$ and the non-negativity constraints $x_1, \ldots, x_m \ge 0$ where $\mathbf{p} \gg \mathbf{0}$ and $m \ge 0$. In other words, we ask whether the standard consumers problem of microeconomics has a solution. Fortunately for us, it does.

Theorem 29.20.1. *Let $u\colon \mathbb{R}^m \to \mathbb{R}$ be a continuous utility function. Suppose $\mathbf{p} \gg \mathbf{0}$ and $m \ge 0$. Then the consumer's utility maximization problem, maximizing utility u over the budget set $B(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}^m_+ : \mathbf{p}\cdot\mathbf{x} \le m\}$ has a solution.*

Proof. Let $B(\mathbf{p}, m)$ denote the budget set. Now $\mathbf{0} \in B(\mathbf{p}, m)$, so the budget set is non-empty. In Example 13.13.2, we found that such budget sets are closed, while Example 29.1.1 showed they are bounded. This means that the budget set is compact.

Weierstrass's Theorem now tells us that the continuous function u can be maximized on $B(\mathbf{p}, m)$, that is, under the constraints given. □

29.21 Utility Maximization can be Impossible

The proof that utility can be maximized relies on a continuous utility function (allowing use of Weierstrass's Theorem) and strictly positive prices (bounding the budget set). If either condition fails—either some price is zero or preferences are not continuous—the consumer's utility maximization problem may not have a solution. The first case is considered in the next example, which sets one of the prices to zero, and finds there is no solution.

► **Example 29.21.1: No Utility Max with Zero Price.** In $\mathbb{R}^2_+$, let utility be $u(x_1, x_2) = x_1 x_2$, $m = 1$ and $\mathbf{p} = (1, 0)$. Here utility is continuous, but one of the prices is zero. The point $(1, n)$ is in the budget set for any n. It yields utility $u(1, n) = n$. There is no utility maximum as taking n sufficiently large would be better than any would-be maximum. ◄

There are times when utility maximization is possible even with a zero price. If we replace the utility function in Example 29.21.1 with $v(x_1, x_2) = x_1 - (x_2 - 10)^2$, there is a utility maximum at $\mathbf{x} = (1, 10)$.

The second case, discontinuous utility, is the subject of the following example. Again, the consumer's utility maximization problem cannot be solved.

► **Example 29.21.2: No Utility Max with Discontinuous Utility.** Again, define utility on $\mathbb{R}^2_+$, this time with

$$u(x_1, x_2) = \begin{cases} x_1 & \text{when } x_2 > 0 \\ 0 & \text{when } x_2 = 0. \end{cases}$$

This utility function is not continuous. Consider the case $m = 1$ and $\mathbf{p} = (1, 1)$. Since $x_1 \leq 1$, we know that the maximum utility can be no more than 1. However, attaining utility 1 requires $x_1 = 1$. The budget constraint becomes $1 + x_2 \leq 1$, implying $x_2 = 0$. This means utility is actually zero.

If we keep $x_2 > 0$, $x_1 = 1 - x_2$, so $u(x_1, x_2) = 1 - x_2 < 1$. By taking x_2 very small, we can get as close to one *util* as we like, but cannot actually attain it. There is no maximum because any utility level less than one can be beaten, while utility one cannot be attained. ◄

The failure of compactness of $B(\mathbf{p}, m)$ or of continuity of preference does not necessarily mean that there will be no solution. There are times when the consumer's utility maximization problem has a solution even though utility is discontinuous and the budget set is not compact.

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