14. Calculus of Several Variables

10/08/20

Although the chapter title refers to calculus, the focus of the chapter is somewhat narrower. We focus on various types of derivatives.

14.1 The Ordinary Derivative

We begin by recalling the ordinary derivative. Let $U \subset \mathbb{R}$ be an open set. A function $f \colon U \to \mathbb{R}$ is *differentiable at* $x_0 \in U$ if

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. It is *differentiable on* U if the *derivative* $f'(x_0)$ exists for every $x_0 \in U$. The expression inside the limit, both here and in similar definitions, is the *difference quotient*. The derivative is also denoted df/dx.

The difference quotients used to define the derivative are the slopes of chord running between $\big(x_0, f(x_0)\big)$ and $\big(x_0 + h, f(x_0 + h)\big)$. We take the limit of these slopes to find the slope of the tangent to f at x_0. The derivative can be used to define a line, L by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

The line L is tangent to the graph of f at the point $\big(x_0, f(x_0)\big)$.

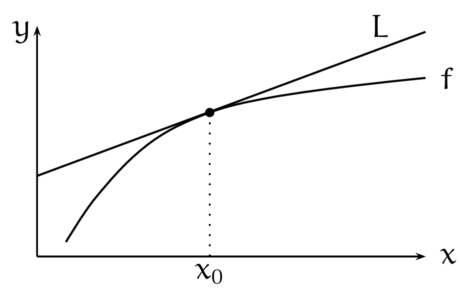

Figure 14.1.1: The tangent line L at x_0 has the equation $y = f(x_0) + f'(x_0)(x - x_0)$.

Linear Approximation. The tangent line L is a linear approximation of $f(x)$ for x near x_0. Setting $\Delta x = x - x_0$, and $\Delta y = y - f(x_0)$, the equation is $y - f(x_0) = f'(x_0)\Delta x$. Now y approximates $f(x)$, so $f(x) \approx f(x_0) + f'(x_0)\Delta x$.

14.2 Partial Derivatives

In economics, we often deal with functions defined on a subset of $\mathbb{R}^m$. Production and utility functions, cost, expenditure, and profit functions are all examples. Let $U \subset \mathbb{R}^m$ be an open set and suppose $f\colon U \to \mathbb{R}$.

We define the i^{th} partial derivative of f at $\mathbf{x}_0 \in U$ by

$$\frac{\partial f}{\partial x_i}(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\boldsymbol{e}_i) - f(\mathbf{x}_0)}{h}$$

provided the limit exists.

In the partial derivative, all of the variables except for x_i are held constant. We then compute the partial derivative by taking the limit of difference quotients as $h \to 0$. It gives the slope of a tangent line in direction $\boldsymbol{e}_i$. It is also the rate of change of the function f in the direction $\boldsymbol{e}_i$.

Since all of the variables except x_i are held constant when computing $\partial f/\partial x_i$, we can compute $\partial f/\partial x_i$ by treating the other variables as constant and taking the ordinary derivative. Thus

$$\begin{aligned}\frac{\partial(x^2 + y^2)}{\partial x} &= 2x,\\ \frac{\partial(x^2y + xyz + y^2z^3)}{\partial x} &= 2xy + yz, \text{ and}\\ \frac{\partial(x^2y + xyz + y^2z^3)}{\partial y} &= x^2 + zy + 2yz^3.\end{aligned}$$

If a partial derivative of f exists, it too may have partial derivatives. Expressions such as

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}.$$

are used to write second (and higher) partial derivatives. Another common notation uses subscripts. Thus

$$\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

$\mathcal{C}^k$ Functions. The notation $\mathcal{C}^k$ denotes the set of k-times continuously differentiable functions. A function $f \in \mathcal{C}^k$ if and only if all of the first k partial derivatives exist and are continuous. When all of the partial derivatives of f exist and are continuous, we write $f \in \mathcal{C}^\infty$. A vector-valued function $\mathbf{f}$ is $\mathcal{C}^k$ if and only if each component function is $\mathcal{C}^k$.

14.3 Examples Using Partial Derivatives

In consumer theory, partial derivatives can be used to compute marginal utilities and marginal rates of substitution.

▶ Example 14.3.1: Utility Functions. Suppose we have a Cobb-Douglas utility function

$$u(x, y) = x^{\alpha} y^{1-\alpha}$$

defined on $\mathbb{R}^2_{++}$ where $0 < \alpha < 1$. The partial derivatives are the marginal utilities

$$MU_x = \frac{\partial u}{\partial x} = \alpha x^{\alpha-1} y^{1-\alpha} \text{ and } MU_y \frac{\partial u}{\partial y} = (1-\alpha) x^{\alpha} y^{-\alpha}.$$

From this we can conclude that the *marginal rate of substitution* is

$$MRS_{xy} = \frac{MU_a}{MU_b} = \frac{\alpha}{1-\alpha}\frac{y}{x}.$$

◀

Partial derivatives are also used in producer theory. For one, they can be used to express marginal products and the marginal rate of technical substitution.

▶ Example 14.3.2: Production Functions. Similarly, if

$$f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$$

is a production function on $\mathbb{R}^2_{++}$, the partial derivatives are the marginal products

$$MP_1 = \frac{\partial f}{\partial x_1} = \frac{1}{2\sqrt{x_1}} \text{ and } MP_2 = \frac{\partial f}{\partial x_2} = \frac{1}{2\sqrt{x_2}}.$$

This means the *marginal rate of technical substitution* is

$$MRTS_{12} = \sqrt{\frac{x_2}{x_1}}.$$

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14.4 Partial Derivatives and Approximation

We can also use partial derivatives to approximate functions. The key is that if we have a small change in x, Δx, the function will change by approximately

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x.$$

Here's an example to show how it works.

▶ Example 14.4.1: Approximation. Take the Cobb-Douglas production function $F(K, L) = 6K^{1/3}L^{2/3}$. Suppose $K = 8000$ and $L = 2744$, yielding $F(8000, 2744) = 23,520$. Now suppose K increases by $\Delta K = 10$. We estimate the effect on output using $\partial f/\partial K$.

$$\frac{\partial f}{\partial K}(8000, 2744) = 2K^{-2/3}L^{2/3} = 2(8000)^{-2/3}(2744)^{2/3} = .98$$

so we expect output to increase by about

$$\frac{\partial f}{\partial K}(8000, 2744) \times \Delta K = 0.98 \times 10 = 9.8.$$

to $23,520 + 9.8 = 23,529.8$. The actual value is $f(8010, 2744) \approx 23,529.796$, so you can see it is a pretty good approximation, at least for relatively small changes in K or L. ◀

14.5 Fréchet Differentiable Functions

The next step is to consider differentiability for vector-valued functions. The definition may look a little strange if you have never seen it before. However, it is a generalization of the ordinary derivative.

Fréchet Derivative. Suppose $\mathbf{f}\colon U \to \mathbb{R}^m$ where U is an open subset of $\mathbb{R}^k$ The function $\mathbf{f}$ is *Fréchet differentiable* at $\mathbf{x}_0 \in U$ if there is a linear function $L\colon \mathbb{R}^k \to \mathbb{R}^m$ such that

$$\lim_{\|\mathbf{h}\|\to 0} \frac{\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - L(\mathbf{h})}{\|\mathbf{h}\|} = 0.$$

In that case L is the *Fréchet derivative* at $\mathbf{x}_0$. If $\mathbf{f}$ is Fréchet differentiable at every $\mathbf{x}_0 \in U$, we say that $\mathbf{f}$ is Fréchet differentiable on U. The Fréchet derivative at $\mathbf{x}_0$ is denoted $D\mathbf{f}(\mathbf{x}_0)$, or $D\mathbf{f}_{\mathbf{x}_0}$. If we need to to specify which variables we are using to differentiate, we will use $D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0)$.

We apply the definition to the dot product.

► **Example 14.5.1: Dot Product.** An easy example is $f(\mathbf{x}) = \mathbf{p}\cdot\mathbf{x} = \sum_{i=1}^{k} p_i x_i$. Here $f\colon \mathbb{R}^k \to \mathbb{R}$. Since it is a linear function, it is its own linear approximation.

If we set $L(\mathbf{h}) = f(\mathbf{h}) = \mathbf{p}\cdot\mathbf{h}$, we find that

$$f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - L(\mathbf{h}) = \mathbf{p}\cdot(\mathbf{x}_0 + \mathbf{h}) - \mathbf{p}\cdot\mathbf{x}_0 - \mathbf{p}\cdot\mathbf{h} = 0.$$

We divide by $\|\mathbf{h}\|$ and take the limit. The limit is zero, so the linear function $L\colon \mathbb{R}^k \to \mathbb{R}$ is the Fréchet derivative. As a linear functional, it is represented by the row vector $\mathbf{p}$. In other words,

$$D(\mathbf{p}\cdot\mathbf{x})_{\mathbf{x}_0} = \mathbf{p}^T = (p_1, \ldots, p_k)$$

for all $\mathbf{x}_0 \in \mathbb{R}^k$. ◄

14.6 Fréchet and Ordinary Derivatives

Suppose $k = m = 1$, so $f\colon \mathbb{R} \to \mathbb{R}$. In this case the Fréchet derivative is the same as the ordinary derivative. One way to see that is to rewrite the definition of the ordinary derivative.

$$\begin{aligned}
f'(x_0) &= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\
0 &= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) \\
&= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - hf'(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - L(h)}{h}
\end{aligned}$$

where $L(h) = f'(x_0)h$ is a linear function of h.

14.7 Matrix Form of the Derivative

We're about ready to write down the general form of the Fréchet derivative for a $\mathcal{C}^1$ function.

When dealing with derivatives of vector functions from $\mathbb{R}^k$ to $\mathbb{R}^m$ where $m > 1$, we will have to be somewhat pedantic about how the vectors are written. Writing them properly allows us to express certain relations as matrix products. Write them wrongly, and you get nonsense. Although we sometimes write a vector $\mathbf{x} \in \mathbb{R}^m$ in casual fashion as $(x_1, \ldots, x_m)$, here it needs to be written as a column

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.$$

When dealing with derivatives, we must use the formal column form. If $\mathbf{f}\colon \mathbb{R}^k \to \mathbb{R}^m$ is a vector function, the derivative must be a linear function from $\mathbb{R}^k \to \mathbb{R}^m$. That means it can be represented by an $m \times k$ matrix with terms

$$\big[D\mathbf{f}(\mathbf{x}_0)\big]_{ij} = \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)$$

Then we write the derivative of

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} \text{ as } D\mathbf{f}(\mathbf{x}_0) = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial x_1/\partial x_k \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial x_2/\partial x_k \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \partial f_m/\partial x_2 & \cdots & \partial x_m/\partial x_k \end{pmatrix}$$

where all of the partial derivatives are evaluated at $\mathbf{x}_0$. The derivative matrix $D\mathbf{f}$ is sometimes called the *Jacobian derivative*, *Jacobian matrix*, or just plain *Jacobian*. The linear function $L(\mathbf{h})$ that the Jacobian represents is formed by multiplying the Jacobian by a vector $\mathbf{h}$, obtaining a vector in $\mathbb{R}^m$.

14.8 The Derivative: An Example

Consider the function $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^2$.

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1 x_2^2 + x_1 x_3 \\ x_2^2 + x_1^2 x_3 + x_2 x_3 \end{pmatrix}.$$

The derivative is a linear mapping from $\mathbb{R}^3 \to \mathbb{R}^2$, and is represented by a 2×3 matrix.

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_2^2 + x_3 & 2x_1 x_2 & x_1 \\ 2x_1 x_3 & 2x_2 + x_3 & x_1^2 + x_2 \end{pmatrix}$$

We evaluate the derivative at $\mathbf{x}_0 = (1, 1, 2)^\mathsf{T}$, obtaining

$$D\mathbf{f}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 4 & 2 \end{pmatrix}.$$

Finally, the linear function $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ is

$$L(\mathbf{h}) = \left[D\mathbf{f}_{\mathbf{x}_0}\right]\mathbf{h} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 4 & 2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

14.9 A Different Matrix Derivatives

Should we have a function $\mathbf{f}\colon \mathbb{R}^k \to \mathbb{R}^m$ that starts as a row vector, $\big(f_1(\mathbf{x}), \dots, f(\mathbf{x}_K)\big)$ we will take the derivative to be $k \times m$ matrix $D\mathbf{f} = (D(\mathbf{f}^\mathsf{T}))^\mathsf{T}$. In other words,

$$D\mathbf{f} = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_m/\partial x_1 \\ \partial f_1/\partial x_2 & \cdots & \partial f_m/\partial x_2 \\ \vdots & \ddots & \vdots \\ \partial f_1/\partial x_k & \cdots & \partial f_m/\partial x_k \end{pmatrix}.$$

In such a case, the linear mapping $L(\mathbf{h})$ is not formed by post-multiplying $D\mathbf{f}$ by $\mathbf{h}$, but by pre-multiplying by $\mathbf{h}^\mathsf{T}$.

When we consider second derivatives of functions from $\mathbb{R}^m \to \mathbb{R}$, the first derivative will be a row vector, and we can use the same method as above to take its derivative that will then be represented by an $m \times m$ matrix. It will define a bilinear functional. One of the vectors will be used to post-multiply D^2f, the other will be transposed so it can pre-multiply D^2f. In combination, this gives us a bilinear form.

14.10 Fréchet Derivatives and Approximation

Examining the definition, we see that the Fréchet derivative $D\mathbf{f}(\mathbf{x}_0)$ defines a linear approximation to the function $\mathbf{f}$ by

$$\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) \approx \mathbf{f}(\mathbf{x}_0) + L(\mathbf{h}) \tag{14.10.1}$$

where $L = D\mathbf{f}(\mathbf{x}_0)$.

To understand the approximation a bit better, we define the *remainder* R by

$$R(\mathbf{x}_0, \mathbf{h}) = f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - L(\mathbf{h}).$$

The remainder is the difference between the function f and its linear approximation $f(\mathbf{x}_0) + L(\mathbf{h})$. By the definition of the derivative,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R(\mathbf{x}_0, \mathbf{h})}{\|\mathbf{h}\|} = 0.$$

This can also be written

$$\frac{R(\mathbf{x}_0, \mathbf{h})}{\|\mathbf{h}\|} = o\big(\|\mathbf{h}\|\big) \text{ at } \mathbf{0}.$$

This tells us that the linear approximation is fairly accurate near $\mathbf{x}_0$, and gets more accurate the closer we are to $\mathbf{x}_0$. In fact, the error term, the remainder, converges to zero enough faster that $\|\mathbf{h}\|$, that the ratio of them also converges to zero.

14.11 Marginal Rate of Substitution: Interpretation

We can use our knowledge of approximation to interpret the marginal rate of substitution. Suppose we start at $\mathbf{x}_0$. Now consider the indifference curve through $\mathbf{x}$, $\{\mathbf{x} : u(\mathbf{x}) = u(\mathbf{x}_0)\}$. Let's make a small change in x_i, holding everything else constant. This moves us off the $u(\mathbf{x})$ indifference curve. Now change x_j to return us the the indifference curve. Since indifference curves slope downward (preferences are monotonic), there is a trade-off between goods i and j. They must have opposite signs.

We can now use the linear approximation. Define $\mathbf{h}$ by $h_i = \Delta x_i$, $h_j = \Delta x_j$, and $h_k = 0$ for $k \neq i, j$, so $\mathbf{h} = \Delta x_i \boldsymbol{e}_i + \Delta x_j \boldsymbol{e}_j$. Then

$$\Delta u \approx Df(\mathbf{x}_0)\mathbf{h} = \frac{\partial u}{\partial x_i}\Delta x_i + \frac{\partial u}{\partial x_j}\Delta x_j = MU_i\,\Delta x_i + MU_j\,\Delta x_j$$

It follows that $MU_i\,\Delta x_i + MU_j\,\Delta x_j \approx 0$, so

$$MRS_{ij} = \frac{MU_i}{MU_j} \approx -\frac{\Delta x_j}{\Delta x_i}$$

In other words, if we plot a slice of the indifference curve in x_i-x_j space, with x_i on the horizontal axis, the marginal rate of substitution approximates the negative slope of the secant through the points $\big(\mathbf{x}_0 + \Delta x, u(\mathbf{x}_0 + \Delta x)\big)$ and $\big(\mathbf{x}_0, u(\mathbf{x}_0)\big)$. Of course, if we let $\Delta x_i, \Delta x_j \to 0$, we obtain the negative slope of the tangent at $u(\mathbf{x}_0)$.

14.12 Total Derivatives and Linear Functionals

When $f\colon \mathbb{R}^k \to \mathbb{R}$, the derivative takes the simpler form

$$Df = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_k}\right) \tag{14.12.2}$$

Sometimes this is written in what appears to be a quite different manner

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \cdots + \frac{\partial f}{\partial x_k}\,dx_k. \tag{14.12.3}$$

They're supposed to both be the derivative. What is going on here?

Recall that the derivative of f is a linear function from $\mathbb{R}^k$ to $\mathbb{R}$. As such, it can properly be written as a covector, a row vector. That is what we have done in equation (14.12.2). But what about equation (14.12.3)?

The obvious interpretation is that we mean that dx_i must the i^{th} basis element $\boldsymbol{e}_i^* = (0, \dots, 0, 1, 0, \dots, 0)$ with a 1 in the i^{th} position.

Another way to see this is to consider the i^{th} coordinate function $x_i(\boldsymbol{x}) = x_i$, which maps $\mathbb{R}^k$ to $\mathbb{R}$. Its derivative, naturally denoted dx_i must be the linear functional $\boldsymbol{e}_i^*$. This maps $\mathbb{R}^k$ to $\mathbb{R}$ such that $dx_i(\boldsymbol{e}_j) = \boldsymbol{e}_i^*(\boldsymbol{e}_j)$ is given by the Kronecker delta δ_{ij}. Either way, the differentials of the such functions form a basis for row vectors.

14.13 One-Forms

The expression used for df in equation (14.12.3) is a simple form of a *differential form*. This type of functional linear combination of the dx_i is a called a 1-*form*. It is important that there are no products of the dx_i terms and all terms have a single dx_i. Real-valued functions are sometimes referred to a 0-forms.

This type of differential form is a linear combination of the basis vectors of $(\mathbb{R}^m)^*$ where the coefficients are functions. In this case the functions are the derivatives $\partial f/\partial x_i$. This type of differential form produces a linear functional on $\mathbb{R}^m$ for every $\mathbf{x}$ In other words, a *differential* 1-*form* is a function whose values are linear functionals.

Because the derivative of f is a linear approximation to f, if we feed $Df(\mathbf{x}_0)$ a vector such as

$$\Delta\mathbf{x} = \Delta\mathbf{x} = \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_m \end{pmatrix} = \sum_i \Delta x_i\, \boldsymbol{e}_i,$$

we obtain

$$\begin{aligned} \Delta f &\approx Df_{\mathbf{x}_0}\, \Delta\mathbf{x} \\ &= \sum_{i=1}^{m} \left[\frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, dx_i \left(\sum_{j=1}^{m} \Delta x_j\, \boldsymbol{e}_j \right) \right] \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, \delta_{ij}\, \Delta x_j \\ &= \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, \Delta x_i \end{aligned}$$

which is a linear approximation to the change in f as $\mathbf{x}_0$ is replaced by $\mathbf{x}_0 + \Delta x$ (see also equation (14.10.1)).

If we used the df version, we would get the same result. So why have two versions? Among other things, differential forms are used in integration. Products there appear in higher derivatives (k-forms) and correspond to area or volume elements. The appropriate product here is something called the **exterior** or **wedge product**, which is both alternating and multilinear, like determinants.

Higher Fréchet derivatives, $D^k f$, are multilinear functions, k-tensors. They are not to be confused with the k-forms used in multi-dimensional integration. The k-forms are alternating, the k^{th} derivatives are not. Quite the contrary, terms such as $\partial^2 f/\partial x_i^2$ appear and at times are important. They would be zero if the tensors were alternating.

14.14 Curves and Tangents

A *curve* in $\mathbb{R}^m$ is an m-tuple of continuous functions

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{pmatrix}$$

where each $x_i : I \to \mathbb{R}$ where I is a open subset of $\mathbb{R}$. The functions are the *coordinate functions* of the curve $\mathbf{x}$ and t is a parameter describing the curve. Curves are allowed to cross or repeat themselves.

One of the simplest curves is a straight line. Recall that straight lines can be written in parametric form as

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{x}_1 \tag{10.30.3.}$$

The curve $\mathbf{x}(t) = (\cos t, \sin t)$ repeatedly traces out a circle of radius one centered at the origin. By adding a coordinate, we obtain $\mathbf{y}(t) = (\cos t, \sin t, t)$, which is a right-handed helix in $\mathbb{R}^3$.

If $\mathbf{x}$ is differentiable, we can define the *tangent vector* as

$$\mathbf{x}'(t) = \left(x_1'(t), \dots, x_m'(t)\right)^{\mathsf{T}}.$$

It describes the instantaneous rate of change of the curve, both direction and magnitude.

When $\mathbf{x}(t)$ is the path of an actual object in motion and t is time, $\mathbf{x}'(t)$ is the velocity at any time t and $\|\mathbf{x}(t)\|$ is the speed. The second derivative $\mathbf{x}''(t) = \left(x_1''(t), \dots, x_m''(t)\right)^{\mathsf{T}}$ is the acceleration at time t. Here $x_i''(t)$ denotes the ordinary second derivative d^2x_i/dt^2.

14.15 Examples of Curves

Our first example is a straight line.

▶ Example 14.15.1: Straight Lines. The straight line $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{x}_1$ has tangent $\mathbf{x}'(t) = \mathbf{x}_1$, the direction of the line. In this form, there is no acceleration, $\mathbf{x}''(t) = \mathbf{0}$. The same line can be traced out in other ways, for example the curve $\mathbf{x}(t) = \mathbf{x}_0 + t^3\mathbf{x}_1$ visits all of the points on that straight line, but does it in a different fashion, with tangent $2t\mathbf{x}_1$. Except at $t = 0$, it points in the same direction, but has a different magnitude, reflecting the variously slower and faster motion along the line. Moreover the acceleration is not zero, but $\mathbf{x}''(t) = 2\mathbf{x}_1$. ◀

The second example circle around and around the unit circle about zero.

▶ Example 14.15.2: Perpetual Circle. The perpetual circle is defined by $\mathbf{x}(t) = (\cos t, \sin t)$. It continually retraces the circle of radius 1 about the origin. Its tangent vector is $\mathbf{x}'(t) = (-\sin t, \cos t)$. You'll notice that for circular motion, the tangent is orthogonal to the curve, $\mathbf{x}'(t)\cdot\mathbf{x}(t) = 0$. The acceleration $\mathbf{x}''(t) = (-\cos t, -\sin t) = -\mathbf{x}(t)$ always points toward the origin. ◀

Our third example is a right-handed helix, meaning the it winds counter-clockwise.

▶ Example 14.15.3: Right-handed Helix. The helix $\mathbf{y}(t) = (\cos t, \sin t, t)$ has tangent $\mathbf{y}'(t) = (-\sin t, \cos t, 1)$. It is not perpendicular to $\mathbf{x}(t)$ due to the x_3 component. The acceleration is $\mathbf{x}''(t) = (-\sin t, -cost, 0)$ and points toward the x_3-axis. This reflects the constant motion along the x_3-axis combined with circular motion about it. ◀

14.16 Regular Curves

We will often require that curves be sufficiently smooth. Such curves are called *regular*.

Regular Curve. A curve $\mathbf{x}(t) = \big(x_1(t), \ldots, x_m(t)\big)^{\mathsf{T}}$ in $\mathbb{R}^m$ is *regular* if each x_i is continuously differentiable in t and $\mathbf{x}'(t) \neq \mathbf{0}$ for all t.

Examples (14.15.1)-(14.15.3) are all regular.

Regularity rules out sudden changes of direction.

► **Example 14.16.1: Curve with a Cusp.** Let $x(t) = t^3$, and $y(t) = t^2$. Of course, $\mathbf{x}'(t) = (3t^2, 2t)^{\mathsf{T}}$. This curve is not regular at 0 because $\mathbf{x}(0) = (0, 0)$. It makes a sudden change of direction there.

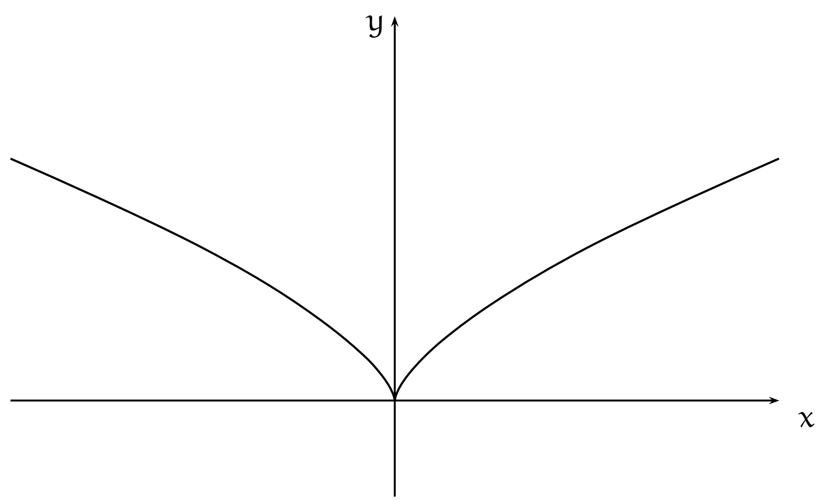

Figure 14.16.2: The point where the curve makes a sudden change of direction is called a *cusp*.

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It's not possible to re-parameterize this curve to make it regular. The tangent changes from going straight down to straight up. Reversing the tangent requires that it be zero at the reversal point, regardless of how the parameter is applied. One way to see this is that if $\mathbf{x}'(t)$ is $\mathcal{C}^1$, so is $x_2'(t)$. For $t < 0$, $x_2'(t)$ is negative, and for $t > 0$, $x_2'(t)$ is positive. Continuity requires that $x_2'(0) = 0$. The only way $\mathbf{x}$ can be regular at zero is if $x_1'(0) \neq 0$. But in that case the tangent could not be straight up or down at $\mathbf{0}$.

14.17 Functions, Curves, and Derivatives

It is sometimes useful to evaluate the derivative of a function f defined along a curve. If $\mathbf{x}(t)$ is a regular curve in $\mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \mathbb{R}$, we can define $g(t) = f \circ \mathbf{x}(t) = f\big(\mathbf{x}(t)\big)$. We can take the derivative of g in the following fashion, based on the chain rule:

$$\begin{aligned} g'(t) &= \frac{\partial f}{\partial x_1}\big(\mathbf{x}(t)\big)x_1'(t) + \frac{\partial f}{\partial x_2}\big(\mathbf{x}(t)\big)x_2'(t) \\ &\qquad + \cdots + \frac{\partial f}{\partial x_m}\big(\mathbf{x}(t)\big)x_m'(t) \\ &= Df_{\mathbf{x}(t)}\,\mathbf{x}'(t). \end{aligned}$$

In the final formula, keep in mind that Df is a $1 \times m$ row vector and $\mathbf{x}'$ is an $m \times 1$ column vector, so the matrix product is a number. Writing the vectors ($\mathbf{x}$) and covectors (Df) in the right way insures everything lines up properly in the product.

As for the formula above, we state the relevant theorem, without proof.

Chain Rule I. *Let $\mathbf{x}(t) = \big(x_1(t), \ldots, x_m(t)\big)$ be a $\mathcal{C}^1$ curve on an interval about t_0 and f a $\mathcal{C}^1$ function defined on a ball in $\mathbb{R}^m$ about $\mathbf{x}(t_0)$. Then $g(t) = f\big(\mathbf{x}(t)\big)$ is a $\mathcal{C}^1$ function on an interval about t_0 and*

$$\begin{aligned} \frac{dg}{dt}(t_0) &= \frac{\partial f}{\partial x_1}\big(\mathbf{x}(t_0)\big)x_1'(t_0) + \cdots + \frac{\partial f}{\partial x_m}\big(\mathbf{x}(t_0)\big)x_m'(t_0) \\ &= Df_{\mathbf{x}(t_0)}\,\mathbf{x}'(t_0). \end{aligned}$$

14.18 Directional Derivatives and Gradients

One way to think about derivatives in a particular direction is to consider a curve that goes in that direction. We want a curve with $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}'(t_0) = \boldsymbol{v}$. There are many such curves, one is the line through $\mathbf{x}_0$ in the direction $\boldsymbol{v}$, given by $\mathbf{x}(t) = \mathbf{x}_0 + (t - t_0)\boldsymbol{v}$.

If $f\colon \mathbb{R}^m \to \mathbb{R}$, we can consider the composite function $g(t) = f\big(\mathbf{x}(t)\big)$ and take its derivative

$$\frac{df}{dt}(t_0) = Df_{\mathbf{x}_0}\,\mathbf{x}'(t_0) = Df_{\mathbf{x}_0}\,\boldsymbol{v}.$$

In fact, Chain Rule I ensures that **any curve** with $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}'(t_0) = \boldsymbol{v}$ will have the **same derivative**. We refer to this as the *directional derivative of* f *in the direction* $\boldsymbol{v}$. Two notations sometimes used for the directional derivative are

$$\frac{\partial f}{\partial \boldsymbol{v}}(\mathbf{x}_0) \quad \text{and} \quad D_{\boldsymbol{v}} f(\mathbf{x}_0)$$

partial Fréchet derivatives taken with respect to a subset of the variables. we prefer the former notation as $D_{\boldsymbol{v}}f$ conflicts with our notation for

It is sometimes useful to regard the derivative of a real-valued function f on $U \subset \mathbb{R}^m$ as a vector rather than a covector. That vector is called the *gradient*, and is written

$$\nabla f(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}_0) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}_0) \\ \vdots \\ \frac{\partial f}{\partial x_m}(\mathbf{x}_0) \end{pmatrix}$$

where $\mathbf{x}_0 \in U$. Of course, $\nabla f(\mathbf{x}_0) = Df(\mathbf{x}_0)^T$. We can then write the directional derivative as

$$\frac{\partial f}{\partial \boldsymbol{v}}(\mathbf{x}_0) = \nabla f(\mathbf{x}_0)\cdot\boldsymbol{v}.$$

14.19 The Chain Rule

The following theorem is called "Chain Rule IV" by Simon and Blume. We'll just call it *the* Chain Rule.

The Chain Rule. *Let U and V be open subsets of $\mathbb{R}^k$ and $\mathbb{R}^\ell$, respectively. Suppose $\mathbf{f}\colon U \to \mathbb{R}^\ell$ and $\mathbf{g}\colon V \to \mathbb{R}^m$ are $\mathcal{C}^1$ functions, and that $\mathbf{x}_0 \in U$ and $\mathbf{y}_0 = f(\mathbf{x}_0) \in V$. We will write $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{y})$.*

Then $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is also $\mathcal{C}^1$ on an open ball about $\mathbf{x}_0$ with

$$D_{\mathbf{x}}\mathbf{h}(\mathbf{x}_0) = D_{\mathbf{y}}\mathbf{g}(\mathbf{y}_0) \times D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0)$$

In other words, the Jacobian derivative of $\mathbf{h}$ is the product of the Jacobian derivatives of $\mathbf{g}$ and $\mathbf{f}$.

The Chain Rule both asserts that the composite function $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is differentiable and gives a formula for its derivative.

We know that $D\mathbf{h}$ is an $k \times \ell$ matrix, and $D\mathbf{f}$ is an $\ell \times m$ matrix, so $D\mathbf{h}$ is an $k \times m$ matrix. Unpacking the matrix product shows that

$$\frac{\partial h_i}{\partial x_j} = \sum_{k=1}^{m} \left(\frac{\partial g_i}{\partial f_k}\right)\left(\frac{\partial f_k}{\partial x_j}\right) \tag{14.19.4}$$

for all $i = 1, \dots, k$ and $j = 1, \dots, m$. Equation (14.19.4) can be written more fully as

$$\frac{\partial h_i}{\partial x_j}(\mathbf{x}_0) = \sum_{k=1}^{m} \left(\frac{\partial g_i}{\partial f_k}\right)(\mathbf{y}_0)\left(\frac{\partial f_k}{\partial x_j}\right)(\mathbf{x}_0),$$

again for all $i = 1, \dots, k$ and $j = 1, \dots, m$. Keep in mind that $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$.

14.20 Chain Rule for Direct and Indirect Variables

The Chain Rule has many applications beyond the obvious ones. Some functions will use the same variables both directly and indirectly via another function.

Consider the function

$$\Phi(\mathbf{x}) = \mathbf{f}\big(\mathbf{x}, g(\mathbf{x})\big).$$

Here $\mathbf{x}$ appears both directly in Φ, and indirectly via g. If $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{f}$ takes $m+1$ arguments. Partition those arguments $(\mathbf{x}, y)$. Define a function $\mathbf{h}\colon \mathbb{R}^m \to \mathbb{R}^{m+1}$ by

$$\mathbf{h}(\mathbf{x}) = \begin{pmatrix} \mathbf{x} \\ g(\mathbf{x}) \end{pmatrix} \quad \text{with} \quad D\mathbf{h}(\mathbf{x}) = \begin{pmatrix} \mathbf{I}_m \\ D_{\mathbf{x}} g \end{pmatrix}$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix.

Then $\Phi(\mathbf{x}) = \mathbf{f}\big(\mathbf{h}(\mathbf{x})\big)$, so the Chain Rule tells us that

$$\begin{aligned} D_{\mathbf{x}}\Phi &= D\mathbf{f}_{(\mathbf{x},y)} \times D_{\mathbf{x}}\mathbf{h} \\ &= D\mathbf{f}_{(\mathbf{x},y)} \times \begin{pmatrix} \mathbf{I}_m \\ D_{\mathbf{x}} g \end{pmatrix} \\ &= D_{\mathbf{x}}\Phi = D_{\mathbf{x}} f + D_y f \times D_{\mathbf{x}} g. \end{aligned}$$

14.21 An Application of the Chain Rule to Integrals

One interesting application of the chain rule is to integrals.

▶ Example 14.21.1: Chain Rule and Integrals. Now consider let $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R}^2 \to \mathbb{R}$ be $\mathcal{C}^1$ and define

$$\Phi(x) = \int_0^{g(x)} h(x,t)\,dt.$$

Now define

$$f(x,y) = \int_0^{y} h(x,t)\,dt$$

and apply the previous result to $\Phi(x) = f\big(x, g(x)\big)$. This yields

$$\Phi'(x) = \frac{d}{dx}\left(\int_0^{g(x)} h(x,t)\,dt\right) = h\big(x, g(x)\big)\,g'(x) + \int_0^{g(x)} \frac{\partial h}{\partial x}(x,t)\,dt.$$

When $g(x) = x$, this reduces to

$$\frac{d}{dx}\left(\int_0^{x} h(x,t)\,dt\right) = h\big(x, g(x)\big) + \int_0^{x} \frac{\partial h}{\partial x}(x,t)\,dt.$$

A similar method can be used on the case where both limits of integration are defined by functions. ◀

14.22 Second Derivatives

10/13/20

Homework: Problems #6, 12, and 17 from Chapter 14, #7 and 24 from Chapter 15 and #12 from Chapter 29 are due on **Tuesday, October 20**.

Second derivatives are a little more complex than first derivatives and we will start with the case where $f\colon \mathbb{R}^m \to \mathbb{R}$ because it is easier to see what is going on. In that case, the first-derivative is a row vector

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f}{\partial x_m}(\mathbf{x}) \end{pmatrix}.$$

This was discussed in section 14.9, where we used a modified matrix representation of such derivatives. We apply that method here.

For functions that are twice continuously differentiable, we write the matrix of second partial derivatives, the *Hessian matrix*, or *Hessian*, as

$$D^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1\,\partial x_2} & \dots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1\,\partial x_m} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2\,\partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \dots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2\,\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_m\,\partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_m\,\partial x_2} & \dots & \frac{\partial^2 f(\mathbf{x})}{\partial x_m^2} \end{pmatrix}.$$

Where

$$\frac{\partial^2 f}{\partial x_k\,\partial x_\ell} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_\ell}\right).$$

14.23 Another Notation

This is one of those times when it is useful to employ an alternate notation for partial derivatives.

$$f_i = \frac{\partial f}{\partial x_i}, \quad f_{ij} = (f_i)_j = \frac{\partial^2 f}{\partial x_j \, \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right), \text{ etc.}$$

Notice the reversal of order between the two notations, ij versus ji. The alternate notation allows us to write the Hessian in the more compact form

$$D^2 f(\mathbf{x}) = \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1m}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2m}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ f_{m1}(\mathbf{x}) & f_{m2}(\mathbf{x}) & \cdots & f_{mm}(\mathbf{x}) \end{pmatrix}.$$

When f is twice continuously differentiable, it does not matter which order is used for the second partial derivatives because

$$\frac{\partial^2 f}{\partial x_k \, \partial x_\ell} = \frac{\partial^2 f}{\partial x_\ell \, \partial x_k}.$$

The derivative is the same either way.[1] If f is not $\mathcal{C}^2$, the order in which we take partial derivatives may affect the result.

[1] In the economics literature, this result is often called Young's Theorem, although a host of mathematicians have worked on the problem, including Cauchy and Lagrange. My experience is that mathematics books usually don't name the theorem after anyone, although I have seen it called the Clairaut-Schwartz Theorem. Clairaut published a proof in 1740 that does not meet modern standards of rigor. The first rigorous proof was due to H.A. Schwartz in 1873. The most commonly used proof is that of Jordan published in 1883. E.W. Hobson and W.H. Young later proved the theorem under weaker conditions, 1907-1909. The name Young's Theorem may derive from the economist R.G.D. Allen.

14.24 The Hessian

So what exactly is the Hessian?

The derivative of $f\colon \mathbb{R}^m \to \mathbb{R}$ is a mapping to a linear function from $\mathbb{R}^m \to \mathbb{R}$. It maps $\mathbf{x}$ to the linear functional $Df(\mathbf{x})$, which takes a single argument $\mathbb{R}^m$. We can write the values of this linear functional for $\mathbf{y} \in \mathbb{R}^m$ as the matrix product $[Df(\mathbf{x})]\mathbf{y}$. The Hessian is a way of writing the derivative of the mapping $\mathbf{x} \mapsto Df(\mathbf{x})$.

The second derivative at $\mathbf{x}$ is a linear function from $\mathbb{R}^m \to (\mathbb{R}^m)^*$. We feed it a (column) vector ($\mathbf{z} \in \mathbb{R}^m$) an get a covector (row vector in $(\mathbb{R}^m)^*$). We can then feed it a second column vector ($\mathbf{y} \in \mathbb{R}^m$), obtaining a number. The Hessian matrix makes it possible to do this. The resulting bilinear map has the form

$$L(\mathbf{y}, \mathbf{z}) = \mathbf{z}^{\mathsf{T}} \left[D^2 f(\mathbf{x})\right] \mathbf{y}.$$

It takes pairs of vectors $(\mathbf{y}, \mathbf{z})$, one to get a linear functional, and a second to produce a number.

Writing this out, we have

$$\begin{aligned} L(\mathbf{y}, \mathbf{z}) &= \mathbf{z}^{\mathsf{T}}\, D^2 f(\mathbf{x})\, \mathbf{y} \\ &= (z_1, \ldots, z_m) \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1m}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2m}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ f_{m1}(\mathbf{x}) & f_{m2}(\mathbf{x}) & \cdots & f_{mm}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \\ &= \sum_{i,j=1}^{m} f_{ij}(\mathbf{x}) z_i y_j. \end{aligned}$$

Since $D^2 f(\mathbf{x})$ is symmetric, it doesn't really matter which vector goes where, but I have written it in the logical order. The vector $\mathbf{y}$ belongs to the linear functional, while the vector $\mathbf{z}$ combines with a linear function that makes to linear functionals.

The resulting mapping from $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is bilinear, and is best thought of as a 2-tensor, a linear mapping from $\mathbb{R}^m \otimes \mathbb{R}^m$ to $\mathbb{R}$. As such we can write

$$D^2 f(\mathbf{x}) = \sum_{ij=1}^{m} f_{ij}(\mathbf{x})\, dx_i \otimes dx_j$$

when we can use the outer product to write

$$L(\mathbf{y}, \mathbf{z}) = \left[D^2 f(\mathbf{x})\right] \cdot (\mathbf{y} \otimes \mathbf{z})$$

The dot product of the two matrices indicates we are multiplying the corresponding term and adding, just like the ordinary dot product, but for matrices. This makes sense because linear functionals on $\mathbb{R}^m$ involve dot products. It shouldn't be surprising to find one here.

You may run across constructions where you vectorize the matrices. This is sometimes done in econometrics. Then taking the dot product of the resulting vectors would give you the same result.

14.25 Taylor's Formula: Preview

One important use of the Hessian is Taylor's formula, which we will prove later.

First, a couple of definitions. In a vector space, the *line segment between x and y* is $\ell(x, y) = \{(1-t)x + ty : 0 \le t \le 1\}$. A set S is *convex* if it contains $\ell(x, y)$ whenever $x, y \in S$.

First-order Taylor's Formula. *Let $f\colon U \to \mathbb{R}$ be $\mathcal{C}^2$ on a convex open set $U \subset \mathbb{R}^m$. Then for every $x, x' \in U$, there is a y on the line segment connecting x and x' such that*

$$f(x') = f(x) + Df(x)(x' - x) + \frac{1}{2}(x' - x)^T \left[D^2 f(y)\right](x' - x). \tag{14.25.5}$$

In Taylor's formula, not only is the Hessian symmetric, but we feed it the same vector on each side, so there is even more symmetry. If $L(\cdot, \cdot)$ is a bilinear form, then $Q(z) = L(z, z)$ is called a *quadratic form*. To see why, write it out. When L is the Hessian, we get

$$\sum_{ij=1}^{m} f_{ij}(x)\, z_i z_j.$$

It is a purely second degree polynomial in the z_i, hence the term quadratic. The Taylor approximation in equation (30.5.2) consists of a constant term, a linear term, and a quadratic term, and for small changes in x, will better approximate f than the derivative alone does.

There are higher Taylor expansions that involve third and higher degree terms, terms which are 3-tensors, 4-tensors, etc.

14.26 Higher Derivatives

Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be k times continuously differentiable with $k > 2$. What do its higher derivatives, those past the Hessian, look like. Well, we know that $D^k f(\mathbf{x})$ must be a k-linear form. That means it can be written as linear map from $(\mathbb{R}^m)^{\otimes k} \to \mathbb{R}$, a k-tensor.

Although perhaps a bit tedious, it's not hard to figure out what $D^k f(\mathbf{x})$ looks like, even though they do not have convenient matrix representations.

$$\big[D^k f(\mathbf{x})\big](\mathbf{y}^1, \ldots, \mathbf{y}^k) = \sum_{i_1,\ldots,i_k=1}^{m} f_{i_1\cdots i_k}(\mathbf{x})\, y^1_{i_1} \cdots y^k_{i_k}$$

where y^i_j is the j^{th} component of $\mathbf{y}^i$. As written here, $y^j_{i_k}$ is the i_k component of $\mathbf{y}^j$. This means that $D^k f(\mathbf{x})$ can be written as the k-tensor

$$D^k f(\mathbf{x}) = \sum_{i_1,\ldots,i_k=1}^{m} f_{i_1\cdots i_k}(\mathbf{x})\, dx_{i_1} \otimes \cdots \otimes dx_{i_k}.$$

30. Calculus of Several Variables II

30.1 Rolle's Theorem

We've already covered Weierstrass's Theorem, which was the first thing in section 30.1 of Simon and Blume. One important consequence of Weierstrass's Theorem is Rolle's Theorem.

Rolle's Theorem. *Suppose* $f\colon [a,b] \to \mathbb{R}$ *is continuous on* $[a,b]$ *and continuously differentiable on* (a,b)*. If* $f(a) = f(b) = 0$*, there is a point* $c \in (a,b)$ *with* $f'(c) = 0$.

Proof. If f is constant on $[a,b]$, then $f(x) = 0$ for all $x \in [a,b]$ and so $f'(x) = 0$ for all $x \in (a,b)$. Then any $c \in (a,b)$ will do.

If f is not constant on $[a,b]$, either there is $d \in (a,b)$ with $f(d) > 0$ or a $d \in (a,b)$ with $f(d) < 0$. In the former case, f has a maximum at some $c \in (a,b)$ by Weierstrass's Theorem. The first-order necessary condition for an interior optimum on $\mathbb{R}$ shows $f'(c) = 0$ (remember your calculus!). In the latter case, f has a minimum at some $c \in (a,b)$ by Weierstrass's Theorem, and so $f'(c) = 0$ by the first-order necessary condition for an interior optimum. Either way, we are done. ∎

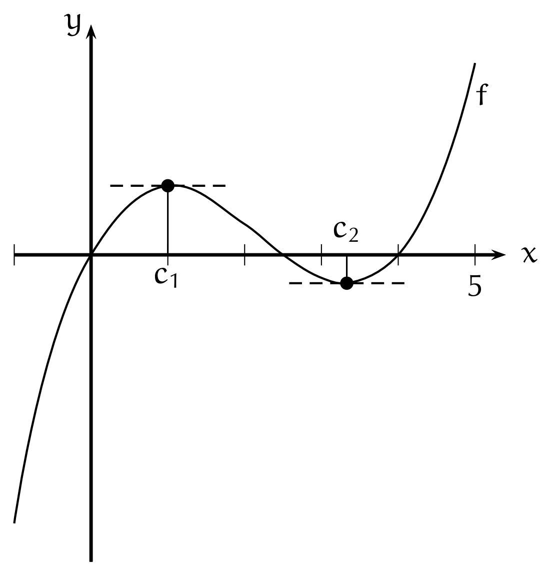

Figure 30.1.1: The function goes both above and below the axes, meaning there must be at least two points in $[0,5]$ that are extrema, obeying $f'(c) = 0$. In this case there are exactly two, labeled c_1 and c_2.

30.2 Mean Value Theorem

The Mean Value Theorem generalizes Rolle's Theorem, which is the key to the proof of the Mean Value Theorem.

Mean Value Theorem. *Let* $f\colon I \to \mathbb{R}$ *be a* $\mathcal{C}^1$ *function on an interval* $I \subset \mathbb{R}$. *Then for any points* $a, b \in I$ *with* $a < b$ *there is a point* c*,* $a < c < b$ *with*

$$f(b) - f(a) = f'(c)(b - a)$$

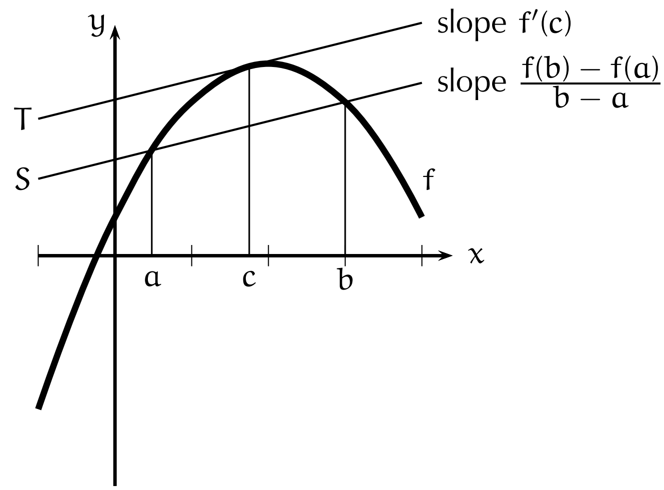

Figure 30.2.1: The Mean Value Theorem gives us a point $c \in (a, b)$ where the slope of the tangent T to f at $(c, f(c))$ is equal to the slope of the secant S through $(a, f(a))$ and $(b, f(b))$.

Proof. Define a function g by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $g(x)$ is the vertical distance between the secant line S and $f(x)$. The distance is zero at both a and b:

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

and

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0.$$

By Rolle's Theorem, there is a $c \in (a, b)$ with

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Then $f'(c)(b - a) = f(b) - f(a)$, proving the result. ∎

30.3 Mean Value Theorem on $\mathbb{R}^m$

There is a version for $\mathbb{R}^m$ that follows directly from the basic Mean Value Theorem.

Theorem 30.3.1. *Let* $f\colon U \to \mathbb{R}$ *be a continuously differentiable function defined on an open set* $U \subset \mathbb{R}^m$. *Suppose* $\mathbf{a}, \mathbf{b} \in U$ *with the line segment* $\ell(\mathbf{a}, \mathbf{b}) \subset U$. *Then there is a point* $\mathbf{c} \in \ell(\mathbf{a}, \mathbf{b})$ *such that*

$$f(\mathbf{b}) - f(\mathbf{a}) = Df_{\mathbf{c}}(\mathbf{b} - \mathbf{a})$$

Proof. Define $g\colon [0,1] \to U$ by $g(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ and set $h(t) = f\big(g(t)\big)$. Then $h\colon [0,1] \to \mathbb{R}$. By the Mean Value Theorem, there is a $t^* \in (0,1)$ with

$$h(1) - h(0) = h'(t^*)(1-0) = h'(t^*).$$

Let $\mathbf{c} = g(t^*) = \mathbf{a} + t^*(\mathbf{b} - \mathbf{a})$. Then

$$\begin{aligned} f(\mathbf{b}) - f(\mathbf{a}) &= h(1) - h(0) \\ &= h'(t^*) \\ &= D_t f\big(g(t^*)\big) \\ &= D_{\mathbf{x}} f\big(g(t^*)\big) g'(t^*) \\ &= Df_{\mathbf{c}}(\mathbf{b} - \mathbf{a}). \end{aligned}$$

∎

This tells us that $f(\mathbf{b}) = f(\mathbf{a}) + Df(\mathbf{c})(\mathbf{b} - \mathbf{a})$ for some $\mathbf{c}$ on the line segment $\ell(\mathbf{b}, \mathbf{a})$, which is useful for approximating f at a point $\mathbf{b}$ based on its value at a point $\mathbf{a}$.

30.4 Taylor Polynomials

There is a generalization of the Mean Value Theorem using Taylor Polynomials. Let $f^{(k)}$ denote the k^{th} derivative of f

$$f^{(k)}(x) = \frac{d^k f}{dx^k}(x)$$

where $f^{(0)}(a) = f(a)$.

Recall that k! denotes k *factorial* which is defined inductively for non-negative integers by $0! = 1$ and $k! = k(k-1)!$. Thus

$$k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k.$$

The gamma function extends the definition to the complex numbers, excepting the non-positive integers. They are related by $k! = \Gamma(k+1)$. When $z \in \mathbb{C}$ has a positive real part,

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx.$$

Taylor Polynomial. The k^{th} order Taylor polynomial is

$$\begin{aligned} P_k(x) &= f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{k!}f^{(k)}(a)(x-a)^k \\ &= \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(a)(x-a)^n. \end{aligned}$$

The fact that $P_k(a) = f(a)$ will be useful when we prove various forms of Taylor's Formula.

Here are the first several Taylor polynomials:

$$\begin{aligned} P_0(x) &= f(x) \\ P_1(x) &= f(x) + f'(a)(x-a) \\ P_2(x) &= f(x) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ P_3(x) &= f(x) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f^{(3)}(a)(x-a)^3 \end{aligned}$$

30.5 First Order Taylor's Formula in $\mathbb{R}$

To see how the proof of Taylor's formula works, we start with the first order Taylor's formula. (The Mean Value Theorem is the zeroth order Taylor's formula.)

Theorem 30.5.1. *Let* $f\colon I \to \mathbb{R}$ *be a* $\mathcal{C}^2$ *function defined on an interval in* $\mathbb{R}$. *If* $a, b \in I$ *there exists a* $c \in (a, b)$ *such that*

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + \frac{1}{2}f''(c)(b-a)^2. \qquad (30.5.1)\\ &= P_1(b) + \frac{1}{2}f''(c)(b-a)^2 \end{aligned}$$

Proof. Define

$$\begin{aligned} g(x) &= f(x) - P_1(x) - M(x-a)^2 \\ &= f(x) - [f(a) + f'(a)(x-a)] - M(x-a)^2. \qquad (30.5.2) \end{aligned}$$

By definition, $g(a) = 0$. Now choose M so that $g(b) = 0$. Then

$$M = \frac{1}{(b-a)^2}\big[f(b) - f(a) - f'(a)(b-a)\big]$$

By Rolle's Theorem, there is a $c_1 \in (a, b)$ with $g'(c_1) = 0$. Now

$$g'(x) = f'(x) - f'(a) - 2M(x-a)$$

so $g'(a) = f'(a) - f'(a) = 0$. Both $g'(a) = g'(c_1) = 0$, so we apply Rolle's Theorem again, this time to g'.

From Rolle's Theorem we obtain a $c \in (a, c_1) \subset (a, b)$ with $g''(c) = 0$. Now

$$g''(x) = f''(x) - 2M$$

so $f''(c) = 2M$. In equation (30.5.2), substitute $M = f''(c)/2$ and set $x = b$ to obtain equation (30.5.1). ∎

30.6 k^{th} Order Taylor's Formula in $\mathbb{R}$

We now consider the k^{th} order Taylor's formula.

Taylor's Formula in $\mathbb{R}$**.** *Let* $f\colon I \to \mathbb{R}$ *be a* $\mathcal{C}^{k+1}$ *function defined on an interval in* $\mathbb{R}$*. If* $a, b \in I$ *there exists a* $c \in (a, b)$ *such that*

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \cdots + \frac{1}{k!}f^{(k)}(a)(b-a)^k \\ &\quad + \frac{1}{(k+1)!}f^{(k+1)}(c)(b-a)^{k+1} \qquad (30.6.3) \\ &= P_k(b) + \frac{1}{(k+1)!}f^{(k+1)}(c)(b-a)^{k+1} \end{aligned}$$

Proof. Define

$$\begin{aligned} g(x) &= f(x) - P_k(x) - M(x-a)^{k+1} \\ &= f(x) - f(a) - f'(a)(x-a) - \frac{1}{2!}f''(a)(x-a)^2 \\ &\quad - \cdots - \frac{1}{k!}f^{(k)}(a)(x-a)^k + M(x-a)^{k+1} \qquad (30.6.4) \end{aligned}$$

where

$$M = \frac{1}{(b-a)^{k+1}}\left[f(b) - P_k(b)\right]$$

As before, $g(a) = f(a) - P_k(a) = f(a) - f(a) = 0$ and M has been chosen so that $g(b) = f(b) - P_k(b) - \left[f(b) - P_k(b)\right] = 0$. By Rolle's Theorem, there is a $c_1 \in (a, b)$ with $g'(c_1) = 0$.

Now

$$\begin{aligned} g'(x) &= f'(x) - f'(a) - f''(a)(x-a) - \cdots - \frac{1}{(k-1)!}f^{(k)}(a)(x-a)^{k-1} \\ &\quad + (k+1)M(x-a)^k. \end{aligned}$$

Then $g'(a) = f'(a) - f'(a) = 0$. A second application of Rolle's Theorem, now to g', yields a $c_2 \in (a, c_1) \subset (a, b)$ with $g''(c_2) = 0$.

(Proof continues on next page...)

30.7 Taylor's Formula Proof Part II

Remainder of Proof.
Computing g'' we obtain

$$g''(x) = f''(x) - f''(a) - \cdots - \frac{1}{(k-2)!} f^{(k)}(a)(x-a)^{k-2} - (k+1)kM(x-a)^{k-1}.$$

It follows that $g''(a) = f''(a) - f''(a) = 0$. A third application of Rolle's Theorem, now to g'', yields a $c_3 \in (a, c_2) \subset (a, b)$ with $g'''(c_3) = 0$.

We continue applying Rolle's Theorem to successive derivatives until we eventually get to $g^{(k)}(x)$ with $g^{(k)}(c_k) = 0$ and

$$g^{(k)}(x) = f^{(k)}(x) - f^{(k)}(a) - (k+1)!\, M(x-a).$$

It follows that $g^{(k)}(a) = f^{(k)}(a) - f^{(k)}(a) = 0$, so we apply Rolle's Theorem one last time to find a $c \in (a, c_k) \subset (a, b)$ with $g^{(k+1)}(c_k) = 0$.

We compute

$$g^{(k+1)}(x) = f^{(k+1)}(x) - (k+1)!\, M.$$

Then $f^{(k+1)}(c) = (k+1)!\, M$, so $M = f^{(k+1)}(c)/(k+1)!$.

In equation (30.6.4), substitute $M = f^{(k+1)}(c)/(k+1)!$ and set $x = b$ to obtain equation (30.6.3). ∎

30.8 Big O, Little o

Big O and *little o* are notations used to describe the asymptotic behavior of functions. They can be applied at infinity, or at any finite point such as 0. The "O" stands for order, and indicates that one function is the same order as the other. They are useful for describing the precision of estimates.[1]

Thus $f(x) = O\big(g(x)\big)$ as $x \to \infty$ means there is an $M > 0$ such that

$$\frac{|f(x)|}{g(x)} \leq M$$

for x large enough.

Similarly $f(x) = O(x^3)$ at 0 means there is an $M > 0$ with

$$\frac{|f(x)|}{x^3} \leq M$$

for x near 0.

Thus

$$10 - \frac{1}{x} = O(1) \quad \text{as } x \to \infty.$$

Big O is used when the ratio of two functions is bounded. We use *little o* when the ratio converges to zero. So $f(x) = o\big(g(x)\big)$ at infinity means that for every $\varepsilon > 0$, there is a K such that

$$\frac{|f(x)|}{g(x)} < \varepsilon$$

for all $x > K$. The definition at any finite point is similar.

When we say $R(x) = o\big(|x|^k\big)$ at $x = 0$, it means that for every $\varepsilon > 0$, there is a $\delta > 0$ with

$$|R(x)| < \varepsilon|x|^k$$

for $|x| < \delta$. In this case f converges to zero enough faster than $|x|^k$ that the ratio also converges to zero.

[1] This type of asymptotic notation is known as the Bachman-Landau notation. Big O was introduced by Paul Bachman in 1894, Edmund Landau proposed the little o notation in 1909. There are other variants that are less commonly used, such as Hardy and Littlewood's Ω notation.

30.9 Big O and little o as Limits

The definitions of big O and little o can also be stated in terms of limits.

Big O. We write $f(x) = O\big(g(x)\big)$ as $x \to a$ to mean

$$\limsup_{x\to a} \frac{|f(x)|}{g(x)} < \infty.$$

The case $a = \pm\infty$ is allowed.

That used the *limit superior*.

Limit Superior. The *limit superior* or lim sup is defined by

$$\limsup_{x\to a} f(x) = \lim_{\varepsilon\to 0} \big(\sup\{f(x) : x \in B_\varepsilon(a)\}\big)$$

when a is finite, and

$$\limsup_{x\to\infty} f(x) = \lim_{k\to\infty} \big(\sup\{f(x) : x > k\}\big)$$

when $a = +\infty$.

The definition is a little simpler for sequences. When we have a sequence $\{a_n\}$,

$$\limsup_{n\to\infty} = \lim_{n\to\infty} \big(\sup\{a_m : m \geq n\}\big).$$

Limit Inferior. The *limit inferior*, lim inf, is defined analogously. E.g.,

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \big(\inf\{a_m : m \geq n\}\big).$$

As for little o:

Little o. We write $f(x) = o\big(g(x)\big)$ as $x \to a$ when

$$\lim_{x\to a} \frac{|f(x)|}{g(x)} = 0.$$

The case $a = \pm\infty$ is allowed.

30.10 The Remainder in Taylor's Formula

Define the k^{th} remainder term by

$$R_k(x; a) = f(x) - P_k(x).$$

We can use Taylor's formula to write

$$\frac{R_k(x; a)}{(x-a)^k} = \frac{1}{k!}\frac{f^{(k+1)}(c)(x-a)^{k+1}}{(x-a)^k} = \frac{1}{k!}f^{(k+1)}(c)(x-a).$$

As $x \to a$, $c \to a$, so the limit of the remainder is

$$\lim_{x \to a} R_k(x; a) = \frac{1}{k!}f^{(k+1)}(a)(a-a) = 0,$$

or in the little o notation,

$$R_k(x; a) = o\big(|x-a|^k\big).$$

This shows that Taylor's formula is a good approximation of f for x near a.

30.11 Example: A Power Series

▶ Example 30.11.1: Power Series. In some cases, the approximation is perfect as $k \to \infty$, and we get a convergent power series. Consider $f(x) = \sin x$ and set $a = 0$. Then $f^{(2k)}(0) = 0$, $f^{(4k+1)}(0) = +1$ and $f^{(4k+3)}(0) = -1$, so the Taylor expansion for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Applying the ratio test, we find that this series converges absolutely for any $x \in \mathbb{R}$ because

$$\left| \frac{x^{2n+3}}{(2n+3)!} \right| \cdot \left| \frac{(2n+1)}{x^{2n+1}} \right| = \left| \frac{x^2}{(2n+2)(2n+3)} \right| \to 0$$

as $n \to \infty$ for every real x.

The remainder obeys

$$|R_{2k+1}| = \frac{|x|^{2k+1}}{k!} \to 0 \quad \text{as } k \to \infty,$$

so the power series converges to $\sin x$. The convergence is uniform on any compact interval. ◀

30.12 Taylor's Formula in $\mathbb{R}^m$

Just as we did with the Mean Value Theorem, we can derive Taylor's Formula in $\mathbb{R}^m$ from the $\mathbb{R}^1$ version. We will use the shorthand notation $\left[D^k f_{\mathbf{a}}\right] \mathbf{h}^{\otimes k}$ to denote the k-tensor $\left[D^k f_{\mathbf{a}}\right](\mathbf{h}, \dots, \mathbf{h})$ that is the k^{th} derivative applied to $\mathbf{h} \otimes \cdots \otimes \mathbf{h} \in (\mathbb{R}^m)^{\otimes k}$.

Taylor's Formula in $\mathbb{R}^m$**.** *Let* $f\colon U \to \mathbb{R}$ *be a* $\mathcal{C}^{k+1}$ *function defined on an open set in* $\mathbb{R}^m$*. Suppose that for every* $\mathbf{a}, \mathbf{b} \in U$*,* $\ell(\mathbf{a}, \mathbf{b}) \subset U$*. Then for all* $\mathbf{a}, \mathbf{b} \in U$ *there exists a* $\mathbf{c} \in \ell(\mathbf{a}, \mathbf{b})$ *such that*

$$\begin{aligned} f(\mathbf{b}) = f(\mathbf{a}) + \left[Df_{\mathbf{a}}\right](\mathbf{b} - \mathbf{a}) + \frac{1}{2!}\left[D^2 f_{\mathbf{a}}\right](\mathbf{b} - \mathbf{a})^{\otimes 2} + \cdots + \frac{1}{k!}\left[D^k f_{\mathbf{a}}\right](\mathbf{b} - \mathbf{a})^{\otimes k} \\ + \frac{1}{(k+1)!}\left[D^{k+1} f_{\mathbf{c}}\right](\mathbf{b} - \mathbf{a})^{\otimes (k+1)} \end{aligned} \tag{30.12.5}$$

Proof. We will piggyback off Taylor's Formula in $\mathbb{R}$. Define

$$\phi(t) = f\big((1-t)\mathbf{a} + t\mathbf{b}\big) = f\big(\mathbf{a} + t(\mathbf{b} - \mathbf{a})\big).$$

Since U is open, $\phi\colon I \to \mathbb{R}$ is a $C^{(k)}$ function defined on an open interval $I \supset [0, 1]$. We can apply Taylor's Formula on the interval I between 0 and 1 to find

$$\phi(1) = \sum_{n=0}^{k} \frac{1}{n!}\left[\phi^{(n)}(0)\right]1^n + \frac{1}{(k+1)!}\left[\phi^{(k+1)}(t^*)\right]1^{k+1}.$$

for some $t^* \in (0, 1)$. Then we apply the Chain Rule to find

$$\begin{aligned} \phi'(t) &= \left[Df\big(\mathbf{a} + t(\mathbf{b} - \mathbf{a})\big)\right](\mathbf{b} - \mathbf{a}), \\ \phi''(t) &= \left[D^2 f\big(\mathbf{a} + t(\mathbf{b} - \mathbf{a})\big)\right](\mathbf{b} - \mathbf{a})^{\otimes 2}, \\ \phi'''(t) &= \left[D^3 f\big(\mathbf{a} + t(\mathbf{b} - \mathbf{a})\big)\right](\mathbf{b} - \mathbf{a})^{\otimes 3} \\ &\text{etc.} \end{aligned}$$

Setting $t = 0$ we obtain equation (30.12.5). ∎

30.13 The Remainder Term in $\mathbb{R}^m$

Again, the k^{th} remainder term is $R_k(x; a) = f(x) - P_k(x)$, so

$$R_k(x; a) = \frac{1}{k!}\left[D^{(k+1)}f_c\right](x - a)^{\otimes(k+1)}.$$

Dividing by $\|x - a\|^k$, we obtain

$$\frac{R_k(x; a)}{\|x - a\|^k} = \frac{1}{k!}\frac{\left[D^{(k+1)}f_c\right](x - a)^{\otimes(k+1)}}{\|x - a\|^k}$$

Let u be the unit vector $(x - a)/\|x - a\|$. Then

$$\frac{R_k(x; a)}{\|x - a\|^k} = \frac{1}{k!}\left[D^{(k+1)}f_c\right]u^{\otimes(k+1)}\|x - a\|$$

As $x \to a$, $c \to a$, and we find

$$\frac{R_k(x; a)}{\|x - a\|^k} \to 0$$

as $x \to a$. Alternatively

$$R_k(x; a) = o\left(\|x - a\|^k\right)$$

as $\|x - a\| \to a$. This means that the remainder goes to zero as $x \to a$ enough faster than $\|x - a\|^k \to 0$ that their ratio converges to zero.

29.3. Connected Sets

Our last collection of important topological concepts relates to connected sets. Roughly speaking, a connected set is a set that is a single contiguous piece. Before defining connectedness, we need another definition.

29.22 Relative Topology

We start by defining the relative or subspace topology.

Relative (Subspace) Topology. Let $(X, \mathcal{T})$ be a topological space and S a subset of X. The *relative* or *subspace topology* on S is defined by $\mathcal{T}_S = \{S \cap U : U \in \mathcal{T}\}$ and $(S, \mathcal{T}_S)$ is called a *subspace of* X.

The relative topology is the weakest topology where the inclusion map $i_S : S \to X$, defined by $i_S(x) = x$, is continuous. In any topology where it is continuous, the sets $U \cap S$ must be open. The relative topology $\mathcal{T}_S$ demands exactly that—no more, no less. The relative topology on S makes S into a topological space (in this case a subspace). When (X, d) is a metric space, the subspace topology is equivalent to (S, d).

Sets of the form $U \cap S$ with U open in X are referred to as *relatively open* and sets of the form $F \cap S$ with F closed in X are called *relatively closed*.

► **Example 29.22.1: Embedding $\mathbb{R}$ in $\mathbb{R}^2$.** When $\mathbb{R}^2$ has the usual topology, we can embed $\mathbb{R}$ into $\mathbb{R}^2$ as $A = \{\mathbf{x} \in \mathbb{R}^2 : x_2 = 0\}$. The relative topology on A is the same as the usual topology on $\mathbb{R}$. This is illustrated in Figure 29.22.2.

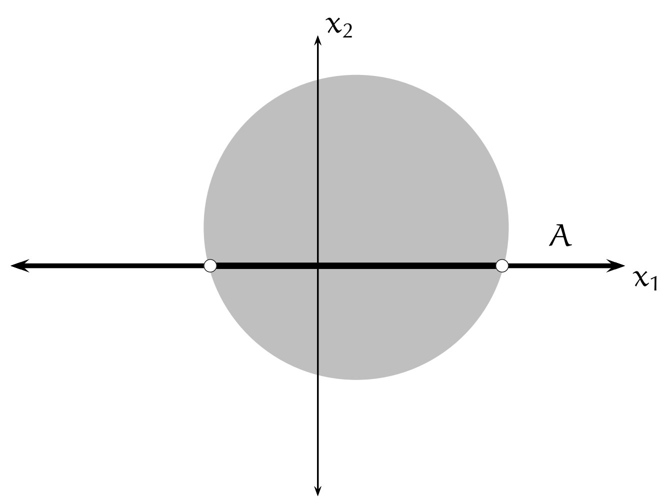

Figure 29.22.2: The intersection of an open ball in $\mathbb{R}^2$ and the line $A = \{\mathbf{x} : x_2 = 0\}$ is an open interval in A. Had we used a closed ball, we would have gotten a closed interval.

◄

29.23 Relative Complements

The *relative complement* of A in S is defined by $S \setminus A = \{x \in S : x \notin A\} = S \cap A^c$.

It's easy to see that the relative complement of a relatively open set is relatively closed and vice-versa. It is easy to show that the relative complements of relatively open sets are relatively closed and vice-versa.

Theorem 29.23.1. *Let $S \subset X$ have the relative topology induced by $(X, \mathfrak{T})$. Then A is the relative complement of an open set if and only if $A = S \cap F$ for some set F that is closed in X.*

Proof. Let $A = S \setminus U = S \cap U^c$ for some open set U. Then $A = S \cap F$ where $F = U^c$ is closed.

Conversely, if $A = S \cap F$ for some closed set F, then $A = S \setminus F^c$. Then $U = F^c$ is open and $A = S \setminus U$. ∎

29.24 Connected Sets

10/15/20

Connected and Disconnected Sets. Let $(X, \mathcal{T})$ be a topological space. The space X is *connected* if there do not exist two disjoint non-empty open sets U and V that obey $X = U \cup V$. If such sets do exist, we say they *disconnect* X and that X is *disconnected*

A subset S of X is *connected* it if is connected in the relative topology.

These definitions can be recast in terms of closed sets. Theorem 29.24.1 applies regardless of whether we are using the base topology on X, or if we are considering a subset with the relative topology.

Theorem 29.24.1. *A space X is disconnected if and only if there are non-empty disjoint closed subsets A and B that cover X.*

Proof. The set X is disconnected if and only if there are non-empty open sets U and V with $U \cap V = \emptyset$ and $U \cup V = X$. Taking complements, and defining the closed sets $A = U^c$ and $B = V^c$, we find this holds if and only if $A \cup B = U^c \cup V^c = X$ and $A \cap B = \emptyset$. Also there are $u \in U$ and $v \in V$, if and only if $u \notin U^c = A$ and $v \notin V^c = B$. Since the sets cover X, that is equivalent to A and B being non-empty. ∎

29.25 Relative Disconnections

Let S have the relative topology and U, V be relatively open sets that disconnect S.

Theorem 29.25.1. *Let $(X, \mathcal{T})$ be a topological space and $S \subset X$. Suppose U and V are relatively open sets that disconnect S. Then there are open sets U' and V' in X obeying*

1. *$U' \cap S$ and $V' \cap S$ are both non-empty.*
2. *$S \subset U' \cup V'$.*
3. *$U' \cap V' \cap S = \emptyset$.*

Conversely, if U' and V' are open sets obeying (1)–(3), $U = U' \cap S$ and $V = V' \cap S$ disconnect S in the relative topology.

Proof. By the definition of the relative topology, there are open sets $U', V' \subset X$ with $U = U' \cap S$ and $V = V' \cap S$. (1) Now $U' \cap S$ and $V' \cap S$ are U and V, which are non-empty. (2) Because U and V are disjoint, $U' \cap V' \cap S = \emptyset$. (3) Finally, U and V cover S, so $(U' \cap S) \cup (V' \cap S) = S$. This follows if $S \subset U' \cup V'$.

We now prove the converse. By (1), $U = U' \cap S$ and $V = V' \cap S$ are are non-empty. By (2), U and V are disjoint. By (3), their union is all of S. Since U and V are also relatively open, they disconnect S. ∎

Notice that it need not be the case that $U' \cap V'$ is empty, only that it contain no points in S.

There's a similar result for closed sets. The proof is quite similar, and has been omitted.

Theorem 29.25.2. *Let $(X, \mathcal{T})$ be a topological space and $S \subset X$. Suppose A and B are relatively closed sets that disconnect S. Then there are closed sets A' and B' in X obeying*

1. *$A' \cap S$ and $B' \cap S$ are both non-empty.*
2. *$S \subset A' \cup B'$.*
3. *$A' \cap B' \cap S = \emptyset$.*

Conversely, if A' and B' are closed sets obeying (1)–(3), $A = A' \cap S$ and $B = B' \cap S$ disconnect S in the relative topology.

Intuitively, a set is connected if there are no breaks in the set, if it is all one piece.

► **Example 29.25.3: A Disconnected Set.** A set such as $S = [0, 1) \cup (1, 2] \subset \mathbb{R}$ is not connected. One disconnection is $A = (-1, 1)$ and $B = (1, 3)$. Both sets are open, both have non-empty intersection with S. Here $A \cap S = [0, 1) \neq \emptyset$ and $B \cap S = (1, 2] \neq \emptyset$. The set S is covered because $A \cup B = (-1, 1) \cup (1, 3) \supset S$. Finally, the sets are disjoint: $A \cap B = \emptyset$. ◄

29.26 Intervals are Connected

In contrast to the previous example, any interval in the real line is a connected set.

Proposition 29.26.1. *Any interval* I *in* $\mathbb{R}$ *is connected.*

Proof. Let I be an interval, possibly infinite. We prove this by contradiction. **Suppose** I **is not connected**, then there are non-empty relatively closed sets A and B that disconnect I. Take $a \in A$ and $b \in B$. We label the sets so that $a < b$ and consider the interval $[a, b]$. Note that $[a, b] \subset I$ because I is an interval and $a, b \in I$.

Define $z = \sup([a, b] \cap A)$. The set $[a, b] \cap A$ is non-empty as $a \in A$. Since the set is bounded, the supremum will exist. By construction, $z \leq b$, so $z \in I$.

Because A and $[a, b]$ are relatively closed, $z \in ([a, b] \cap A) \subset A$. Of course $z \neq b$ because $A \cap B$ is empty. Moreover, z is the upper bound of $A \cap [a, b]$, so any $w \in (z, b]$ must be in B. In particular, $z + \frac{1}{n} \in B$ for n large enough. Letting $n \to \infty$, we find $z \in B$ since B is relatively closed and $z \in I$. But $z \in A$ so it cannot also be in B. **This contradiction** means that there are **no sets** A **and** B **that disconnect** I. Therefore, I is connected. ∎

29.27 Totally Disconnected Sets

Totally disconnected sets are the opposite extreme from connected sets. They contain no connected subsets bigger than a single point. A set is *totally disconnected* if its only connected subsets are the trivial ones—singletons and the empty set. Equivalently, a set is totally disconnected if you can disconnect any two distinct points in the set.

► Example 29.27.1: The Rationals are Totally Disconnected. The rational numbers $\mathbb{Q}$ provide an example of total disconnection. Suppose a is an irrational number (e.g., $\sqrt{2}$, π). Then $A = (-\infty, a] \cap \mathbb{Q}$ and $B = \mathbb{Q} \cap [a, +\infty)$ are relatively closed sets in $\mathbb{Q}$ that disconnect $\mathbb{Q}$. Moreover, you can disconnect any subset of $\mathbb{Q}$ that contains at least two distinct points in the same fashion. That means that the set $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected. ◄

Another totally disconnected set is the Cantor set of Example 12.20.1.

► Example 29.27.2: The Cantor Set is Totally Disconnected. Suppose $x < y$ are distinct elements of the Cantor set. As we saw, both can be written as ternary numbers consisting entirely of 0's and 1's. Take the first digit that differs, call it digit k. Digit k is 1 in x, 3 in y. Let z have ternary expansion identical to x, except that the k^{th} digit is 2. Then $x < z$ and $y > z$. Let $A = \mathfrak{C} \cap (-\infty, z]$ and $B = \mathfrak{C} \cap [z, +\infty)$. Then A and B are closed sets that disconnect $\mathfrak{C}$. ◄

29.28 Connected Components of Disconnected Sets

Let $(X, \mathcal{T})$ be a topological space that is not connected. If a subspace S is connected, we can ask it if is there are any larger connected subspaces that include it. If there are not, then we call S a *connected component* or just a *component* of X.

Given a connected subset S, a maximal connected subset containing S can be constructed by taking the union of all connected subsets of X that contain S. The key step in showing this is that the union of two connected subsets containing S is also connected. In fact, we can show more than we need. If two connected subsets share even a single point, their union is connected.

Theorem 29.28.1. *Suppose* A *and* B *are connected subsets of* $(X, \mathcal{T})$ *containing a point* x. *Then* $A \cup B$ *is connected.*

Proof. **Suppose** that $A \cup B$ **can be disconnected** by relatively open sets U and V. Label the sets U and V so $x \in U$. Keep in mind that $x \in A \cap B$ and take $y \in V \cap (A \cup B)$. There are two possibilities: $y \in A$ or $y \in B$.

Suppose $y \in A$. But then both $A \cap U$ and $A \cap V$ are non-empty, so U and V disconnect A, **which is impossible**.

Otherwise, $y \in B$. Now $x \in U$ and $y \in V$, so both $B \cap U$ and $B \cap V$ are non-empty. Then U and V disconnect the connected set B, **which is also impossible**.

As both possibilites are impossible, we cannot disconnect $A \cup B$. ∎

Earlier we considered the space $[0, 1) \cup (1, 2]$. It has two connected components, $[0, 1)$ and $(1, 2]$. Both sets are connected because they are intervals, and we add even a single point of one interval to the other, we get a disconnected set.

29.29 Continuity and Connectedness

Our next result is that the continuous image of a connected set is connected. This means that we can make connected sets from other connected sets by applying continuous functions to them.

Theorem 29.29.1. *Suppose* $f\colon X \to Y$ *is continuous and* $S \subset X$ *is connected. Then* $f(S)$ *is connected.*

Proof. Suppose not. Let A, B be relatively closed sets that disconnect $f(S)$. Now $f^{-1}(A)$ and $f^{-1}(B)$ are relatively closed because f is continuous. Moreover, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset$ and $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \supset f^{-1}(f(S)) = S$. Thus $f^{-1}(A)$ and $f^{-1}(B)$ disconnect S. As this is impossible, $f(S)$ cannot be disconnected. ∎

An important consequence of this is the Intermediate Value Theorem, which says that if a continuous function defined on an interval takes two values, it also takes all values in between.

Intermediate Value Theorem. *If* $f\colon [a,b] \to \mathbb{R}$ *is continuous, and* y *is a number between* $f(a)$ *and* $f(b)$*, then there is at least one* $c \in [a,b]$ *with* $f(c) = y$.

Proof. We may assume $f(a) < f(b)$ without loss of generality. We proceed by contradiction. If no such c exists, $(-\infty, y]$ and $[y, +\infty)$ disconnect $f([a,b])$. This is impossible by the previous proposition, and so such a c must exist. ∎

A function $f\colon X \to X$ has a *fixed point* if there is an $x^* \in X$ with $f(x^*) = x^*$. The Intermediate Value Theorem can be used to show that any continuous function mapping the unit interval $[0,1]$ to itself has a fixed point.

Theorem 29.29.2. *Let* $f\colon [0,1] \to [0,1]$ *be a continuous function. Then there is* $x^* \in [0,1]$ *with* $f(x^*) = x^*$.

Proof. If $f(0) = 0$ or $f(1) = 1$, we are done.

Otherwise, define $g(x) = f(x) - x$. Then g is continuous with $g(0) > 0$ and $g(1) < 0$. By the Intermediate Value Theorem, there is a x^*, $0 < x^* < 1$, with $g(x^*) = 0$. But then $f(x^*) = x^*$ and we are done. ∎

29.30 Contraction Mapping Theorem

We temporarily put aside connected sets to prove a stronger fixed point theorem, Banach's Contraction Mapping Theorem. This powerful theorem can be used to prove the Inverse and Implicit Function Theorems of S&B Chapter 15. It can be used to show the existence of solutions to differential equations. It has economic applications, including finding solutions to Bellman's dynamic programming equation.

Let (X, d) be a metric space. A function $f\colon X \to X$ is called a *contraction* if there is an $r < 1$ with $d\big(f(x), f(y)\big) \le r\, d(x, y)$ for every $x, y \in X$.

Contraction Mapping Theeorem. *Let f be a contraction on a complete metric space (X, d). Then f has a unique fixed point x^*. Moreover, take $x_0 \in X$ and define $x_n = f(x_{n-1})$ for $n = 1, 2, \ldots$. Then $x_n \to x^*$.*

Proof. First we show uniqueness. Suppose there are two fixed points x^* and y^*. Then

$$d(x^*, y^*) = d\big(f(x^*), f(y^*)\big) \le r\, d(x^*, y^*).$$

This implies $d(x^*, y^*) = 0$, so $x^* = y^*$.

Consider the sequence $\{x_n\}$ given in the statement of the theorem. I claim that $d(x_{n+1}, x_n) \le r^n d(x_1, x_0)$. We prove this by induction. It is true for $n = 1$ because

$$d(x_2, x_1) = d\big(f(x_1), f(x_0)\big) \le r\, d(x_1, x_0).$$

Also, if it is true for n,

$$d(x_{n+2}, x_{n+1}) = d\big(f(x_{n+1}, f(x_n)\big) \le r\, d(x_{n+1}, x_n) \le r^{n+1} d(x_1, x_0)$$

shows it is true for $n + 1$. It follows that it is true for all $n = 1, 2, \ldots$ by induction.

Suppose $m \ge n$. Then

$$d(x_m, x_n) \le \sum_{i=0}^{m-n-1} d(x_{n+i+1}, x_{n+i}) \le \sum_{i=0}^{m-n-1} r^{n+i} d(x_1, x_0) = \frac{r^n}{1-r}.$$

This shows that $\{x_n\}$ is a Cauchy sequence. By completeness of X the sequence has a limit x^*. The function f is continuous, so

$$f(x^*) = f(\lim_n x_n) = \lim_n f(x_n) = \lim x_n = x^*$$

showing that x^* is the unique fixed point of f. ∎

29.31 Path-Connected Sets

It is often easier to show a set is connected by showing it is path-connected. A *path* from a to b in S is a continuous function $f\colon [0,1] \to S$ with $f(0) = a$ and $f(1) = b$.

Path-connected. A set S is *path-connected* if for every $a, b \in S$ there is a path from a to b in S.

Path-connected sets are connected.

Proposition 29.31.1. *Any path-connected set is connected.*

Proof. Suppose S is path-connected and A, B are closed sets with S so that $S \subset A \cup B$. Choose $a \in A \cap S$ and $b \in B \cap S$ and let f be a path between them in S.

Now consider $f^{-1}(A)$ and $f^{-1}(B)$. These are closed sets in $[0,1]$ with $0 \in f^{-1}(A)$ and $1 \in f^{-1}(B)$. Moreover, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \supset [0,1]$. Since $[0,1]$ is connected, there is some $x \in f^{-1}(A) \cap f{-}1(B)$. It follows that $f(x) \in A \cap B$ and that A and B cannot disconnect S. Thus S is connected. ∎

In some spaces the connected and path-connected sets are the same. This happens in $\mathbb{R}$, where the only connected sets are the intervals. This includes trivial intervals such as $[a, a]$ and infinite intervals like $(0, +\infty)$. Intervals are also path-connected, meaning that the set of connected subsets and the set of path-connected subsets are identical in $\mathbb{R}$.

29.32 Convex and Star-shaped Sets are Connected

A convex set is one that contains the line segment between any pair of its points.

Convex Set. Let V be a vector space. A set $S \subset V$ is *convex* if for every $\mathbf{x}, \mathbf{y} \in S$, $\ell(\mathbf{x}, \mathbf{y}) = \{(1-t)\mathbf{x} + t\mathbf{y} : 0 \le t \le 1\} \subset S$.

An immediate corollary is that convex sets are connected.

Corollary 29.32.1. *Any convex set is connected.*

Proof. If S is convex and $x, y \in S$, the function $f(t) = (1-t)x + ty$ is a path from x to y in S.

Star-shaped Set. A set $S \subset \mathbb{R}^m$ is *star-shaped* if there is a point $\mathbf{x}_0 \in S$ so that for any $\mathbf{x}$, $\ell(\mathbf{x}_0, \mathbf{x}) \subset S$.

Star-shaped sets are also connected.

Corollary 29.32.2. *Any star-shaped set is connected.*

Proof. Given $\mathbf{x}, \mathbf{y} \in S$, the path $f(t) = (1-2t)\mathbf{x} + 2t\mathbf{x}_0$ for $t \in [0, 1/2]$ and $f(t) = (2-2t)\mathbf{x}_0 + (2t-1)\mathbf{y}$ for $t \in [1/2, 1]$ is a path from $\mathbf{x}$ to $\mathbf{y}$.

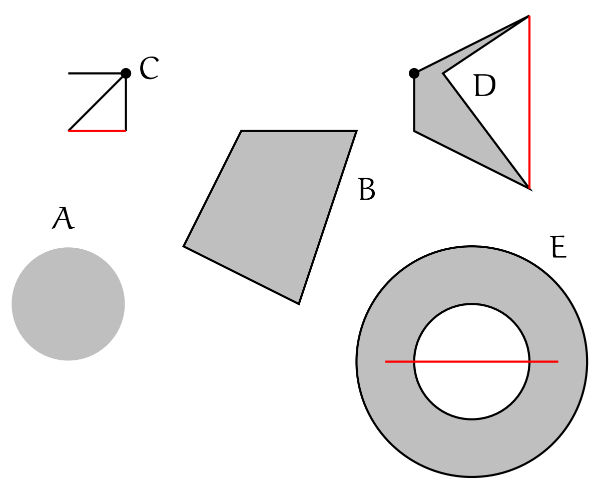

Figure 29.32.3: The sets A and B are convex. The sets C and D are not convex, as demonstrated by the red line segments that must leave the set to connect points. The sets C and D are star-shaped with respect to the heavy dots. The annulus E is neither convex nor star-shaped, although it is connected.

29.33 Example: Connected, But Not Path-Connected

Path-connectedness is quite useful because it is often easier to show a set is connected by showing it is path-connected rather than trying to directly use the definition of connectedness. There are limitations to this approach because the converse is not true. A set can be connected without being path-connected.

► Example 29.33.1: Topologist's Sine Curve. Consider $\mathbb{R}^2$ and let $S = \{(0, y) : -1 \leq y \leq 1\} \cup \{(x, \sin(1/x) : x > 0\}$. This is a variant of the topologist's sine curve of Example 13.4.2. The set S is connected but not path-connected.

Suppose, by way of contradiction, **that** S **is path-connected**. Then there is a continuous path in S, $f(t) = \big(x(t), y(t)\big)$, with $f(0) = (0, 0)$ and $f(1) = (1/\pi, 0)$. The components of f, x and y, inherit continuity from f.

By the Intermediate Value Theorem, there is a t_1, $0 < t_1 < 1$ with $x(t_1) = 2/3\pi$. Then there is a t_2, $0 < t_2 < t_1$ with $x(t_2) = 2/5\pi$. Continue this process to obtain a decreasing sequence $\{t_n\}$ with $t_n \to 0$ and $x(t_n) = 2/(2n + 1)\pi$. Since t_n is a decreasing sequence that is bounded below, it converges to some t_0.

By continuity, $y(t_n) \to y(t_0)$. But this is impossible because $y(t_n) = +1$ when n is even and -1 when n is odd. **This contradiction shows there is no such path** f. The set S cannot be path connected.

It is easy to see that the portion of S with $x > 0$ is path-connected, as is $\{(0, y) : -1 \leq y \leq 1\}$. This means that the only possible disconnection is into $A = \{(0, y) : -1 \leq y \leq 1\}$ and $B = \{(x, \sin(1/x) : x > 0\}$. This fails because B is not relatively closed. It does not contain $\lim_n (1/2\pi n, 0) = (0, 0)$. Therefore S is connected. ◄

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