# 25. Ordinary Differential Equations: Systems of Equations

We now turn to the analysis of systems of equations. As we saw in section 24.14, it is possible to convert a second order differential equation into a first order system of two equations.

Let's start with a general first order linear system of $m$ equations relating relating functions $y_1, \ldots, y_m$ of a variable $t$. We can write the system

$$\begin{aligned}
\dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1m}y_m \\
\dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2m}y_m \\
\vdots &= \quad \vdots \\
\dot{y}_m &= a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mm}y_m
\end{aligned}$$

Setting $\mathbf{y} = (y_1, \ldots, y_m)^\mathbf{T}$, and $\mathbf{A} = [a_{ij}]$, we can rewrite the equation in vector/matrix form as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}.$$

Given initial data of the form $y_i(t_0) = y_i$, define $\mathbf{y}_0 = (y_1, \ldots, y_m)$. We can write the initial value problem as

$$\begin{aligned}
\dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} \\
\mathbf{y}(t_0) &= \mathbf{y}_0
\end{aligned} \tag{25.0.1}$$

We have complete solutions to such initial value problems. The solution has matrix form

$$\mathbf{y}(t) = e^{(t-t_0)\mathbf{A}}\mathbf{y}_0.$$

This implies that each of the $y_i(t)$ can be written as a linear combination of $e^{\lambda_i t}$ for $\lambda_i \in \sigma(\mathbf{A})$ if the eigenvectors of $\mathbf{A}$ span $\mathbb{R}^m$, and terms of the form $t^k e^{\lambda_i t}$ for eigenvalues with algebraic multiplicity larger than their nullity.

## 25.1 More on Solving First Order Linear Systems

If the eigenvalues of $\mathbf{A}$ are all distinct, or if they all have equal algebraic and geometric multiplicity, there is a basis $\mathcal{P}$ consisting of eigenvectors with basis matrix $\mathbf{P}$ where the linear transformation defined by $\mathbf{A}$ takes the diagonal form $\mathbf{D}$ where the diagonal consists of eigenvalues, $\lambda_1, \ldots, \lambda_m$. The matrices $\mathbf{A}$ and $\mathbf{D}$ are related by the change of basis formula $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. If an eigenvalue has multiplicity $m_k$, we list it $m_k$ times. Then

$$e^{(t-t_0)\mathbf{A}} = \mathbf{P}\big(e^{(t-t_0)\mathbf{D}}\big)\mathbf{P}^{-1}.$$

It follows that

$$e^{(t-t_0)\mathbf{A}} = \mathbf{P}\begin{pmatrix} e^{\lambda_1(t-t_0)t} & & \\ & \ddots & \\ & & e^{\lambda_m(t-t_0)t} \end{pmatrix}\mathbf{P}^{-1}$$

In fact, we can write $\boldsymbol{y}_0$ as a sum of eigenvectors

$$\boldsymbol{y}_0 = \sum_{i=1}^{m} c_i \boldsymbol{v}_i$$

when

$$\boldsymbol{y}(t) = \sum_{i=1}^{m} c_i e^{\lambda_i(t-t_0)} \boldsymbol{v}_i.$$

Otherwise, we must use the Jordan canonical form. When unpacking it, this involves terms of using $e^{\lambda_i(t-t_0)}, te^{\lambda_i(t-t_0)}, \ldots, t^k e^{\lambda_i(t-t_0)}$ for a Jordan chain of order $k$ belonging to eigenvalue $\lambda_i$.

## 25.2 $m^{th}$ Order Equations as Matrix Equations

Any $m^{th}$ order linear differential equation can written as a first order equation on $\mathbb{R}^m$. Suppose we have the $m^{th}$ order linear ordinary differential equation

$$a_0 \frac{d^m y}{dt^m} + a_1 \frac{d^{m-1} y}{dt^m} + \cdots + a_{m-1} \frac{dy}{dt} + a_m y = 0. \tag{25.2.2}$$

Because it is $m^{th}$ order, we must have $a_0 \neq 0$. Now define the matrix $\boldsymbol{A}$ by

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{a_m}{a_0} & -\frac{a_{m-1}}{a_0} & \cdots & & -\frac{a_1}{a_0} \end{pmatrix}$$

Then taking $y_1 = y$, we have $\dot{y} = \dot{y}_1 = y_2$, $\ddot{y} = \dot{y}_2 = y_3, \ldots, d^{m-1}y/dt^{m-1} = \dot{y}_{m-1} = y_m$, and

$$\begin{aligned} \dot{y}_m &= d^m y/dt^m \\ &= -\frac{a_m}{a_0} y_1 - \frac{a_{m-1}}{a_0} y_2 - \cdots - \frac{a_1}{a_0} y_m \\ &= -\frac{a_m}{a_0} y - \frac{a_{m-1}}{a_0} \frac{dy}{dt} - \cdots - \frac{a_1}{a_0} \frac{d^{m-1}}{dt^{m-1}}. \end{aligned}$$

Rearranging and multiplying by $a_0$ shows that the matrix equation implies equation (25.2.2).

For example, if

$$2\dddot{y} + 5\ddot{y} + 6\dot{y} + 16y = 0$$

the matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -3 & -5/2 \end{pmatrix}$$

puts the equation into matrix form.

## 25.3 First Order Systems vs. $m^{th}$ Order Equations

Are systems of $m$ equations the same as $m^{th}$ order equations of one variable? Not exactly. Although every $m^{th}$ order linear equation defines an equivalent first order system in $\mathbb{R}^m$, the converse is false. It is possible to write uncoupled equations in matrix form, but not as a single equation.

► Example 25.3.1: Uncoupled System. For example, consider the uncoupled system.

$$\begin{aligned}\dot{y}_1 &= 2y_1\\ \dot{y}_2 &= y_2\end{aligned}$$

The general solution is $y_1 = c_1 e^{2t}$ and $y_2 = c_2 e^t$, and there is no relation between what happens with $y_1$ and $y_2$. In particular, we cannot use information about $y_1$ and $\dot{y}_1$ to find out anything about $y_2$. ◄

This may seem a bit odd, as our solution method involves changing coordinates to the eigenvector system, where the system will usually take this form. However, this change of basis often means that returning to the standard basis will mix the coordinates. In the standard basis, one of the variables may contain information about all of the others. This is always the case when the matrix form is derived from a single equation.

► Example 25.3.2: Reducible System. For comparison, here is a system in $\mathbb{R}^m$ that can be reduced to an equation of order $m$ (or lower). For example, take

$$\begin{aligned}\dot{y}_1 &= ay_2\\ \dot{y}_2 &= -by_1,\end{aligned}$$

where $a, b > 0$. Notice that $\dot{y}_1$ depends on $y_2$ and $\dot{y}_2$ depends on $y_1$.

We can write this as a single equation. Begin by setting $y = y_1$. Now $\dot{y} = ay_2$. We take the derivative, yielding $\ddot{y} = a\dot{y}_2$. The second equation is $\dot{y}_2 = -by$, allowing us to combine them into a single second order equation

$$\ddot{y} = a\dot{y}_2 = a(-by) = -aby.$$

The solution is

$$y_1(t) = y(t) = c_1 \cos\left((ab)^{1/2}t\right) + c_2 \sin\left((ab)^{1/2}t\right).$$

Here knowledge of $y_1(0) = c_1$ and $\dot{y}_1(0) = c_2(ab)^{1/2}$ entirely determine $y_2 = (1/a)\dot{y}_1$. ◄

## 25.4 Vector Fields

A *vector field on* $\mathbb{R}^m$ is a mapping $\mathbf{F}\colon \mathbb{R}^m \to \mathbb{R}^m$ that assigns a vector in $\mathbb{R}^m$ to any point in $\mathbb{R}^m$.

If $\mathbf{A}$ is an $m \times m$ matrix, we can define a vector field on $\mathbb{R}^m$ by $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Many other vector fields are possible, such as

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x_1^2 + \sin x_2 \\ x_1 x_3 + e^{x_1^2 + x_2^2} \\ x_2 - x_3 \end{pmatrix}$$

Another way to define a vector field is to take the gradient of any differentiable function $\phi$. Just set $\mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x})$. This method is often used in physics where $\phi$ is potential energy and $\nabla(-\phi)$ is the corresponding force field. Under Newtonian gravity, the potential energy from a mass $M$ located at the origin is $\phi(\mathbf{x}) = -GM/\|\mathbf{x}\|_2$ where $G$ is the gravitational constant. Then $\nabla(-\phi)$ is the gravitational field, indicating that the gravitational force on a mass $m$ at $\mathbf{x}$ is

$$-m\nabla\phi(\mathbf{x}) = -\frac{GMm}{\|\mathbf{x}\|_2^2}\frac{\mathbf{x}}{\|\mathbf{x}\|_2} = -\frac{GMm}{\|\mathbf{x}\|_2^2}\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ is the unit vector in the $\mathbf{x}$ direction. As Isaac Newton discovered, the gravitational force points toward the mass $M$ at the origin and has magnitude proportional to the inverse square of the distance from the origin.

More generally, vector fields make sense on any differentiable manifold. The only change is that they map points $\mathbf{x} \in M$ to the tangent space of $M$ at $\mathbf{x}$. We define the *tangent bundle of a $\mathcal{C}^1$ manifold* $M$ by $TM = \{(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in T_{\mathbf{x}}M\}$.

**Vector Field.** A *vector field on a $\mathcal{C}^1$ manifold* $M$ is a mapping $\mathbf{F}\colon M \to TM$ such that $\mathbf{F}(\mathbf{x}) \in T_{\mathbf{x}}M$ for all $\mathbf{x} \in M$.

When $M = \mathbb{R}^m$, the tangent space at $\mathbf{x}$, $T_{\mathbf{x}}(\mathbb{R}^m) = \mathbb{R}^m$. As a result, the manifold definition of vector field generalizes the previous definition because when $M = \mathbb{R}^m$, $T_{\mathbf{x}}M = \mathbb{R}^m$ for all $\mathbf{x} \in \mathbb{R}^m$. It follows that a vector field on $\mathbb{R}^m$ (as a manifold) is just a mapping $\mathbf{F}\colon \mathbb{R}^m \to \mathbb{R}^m$, which was our original definition.

## 25.5 Differential Systems and Vector Fields

Vector fields are of interest to us because any first order differential system $\dot{y} = F(y)$ on $\mathbb{R}^m$ defines an associated vector field $F$. The point is that since $F\colon \mathbb{R}^m \to \mathbb{R}^m$, $F$ can be interpreted as a vector field.

If

$$F(y) = \begin{pmatrix} y_1 \\ y_2^2/2 \end{pmatrix}$$

we can represent the vector field by the following diagram.

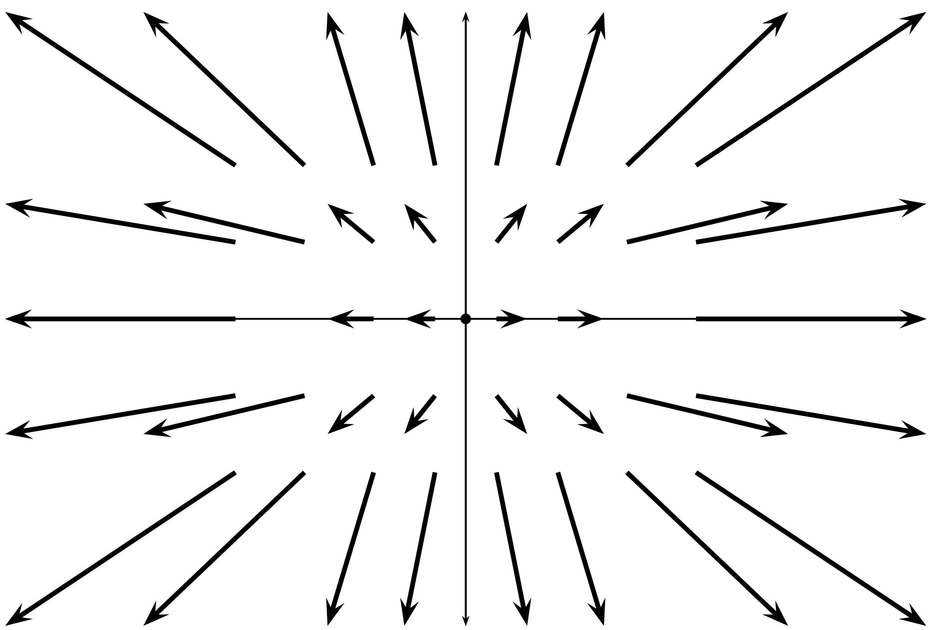

**Figure 25.5.1**: The vector field $F$ is plotted by plotting the vector from $x$ to $x + F(x)$ for various $x$. At the origin, the vector field is zero. We indicate that with a dot at the origin. The length of the arrows increases away from the origin, indicating that motion accelerates as you move away from the origin.

## 25.6 Phase Portraits

An *integral curve* or *trajectory* of $\mathbf{F}$ is the set of points $\{\mathbf{y}(t)\}$ solving $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y})$ for some initial values $\mathbf{y}_0$.

The Peano Existence Theorem gives conditions showing that integral curves exist over a small time interval. If the Picard-Lindelöf Theorem holds, they will be locally unique. In many cases of interest, we are able to write integral curves over a substantial domain. In the case of a linear system, that domain is often $(t_0, +\infty)$ or even $(-\infty, +\infty)$. A plot of sample trajectories is called a *phase portrait*.

Of course, the derivative of an integral curve defines the vector field $\mathbf{F}$.

Suppose we take the equation $\dot{\mathbf{y}} = -2\mathbf{y}$. The solutions are $\mathbf{y}(t) = (c_1e^{-2t}, c_2e^{-2t})$. If $c_1 \neq 0$, we can write $y_2(t)/y_1(t) = c_2/c_1$, which shows that the trajectories are rays about the origin. This yields the following phase portrait.

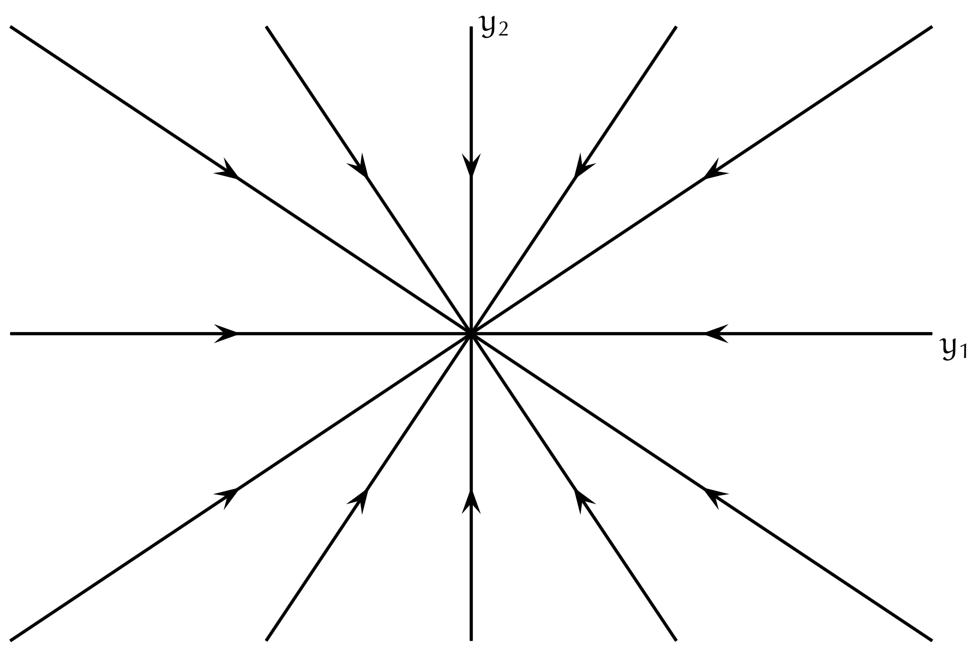


**Figure 25.6.1**: The phase portrait plots trajectories of the form $\mathbf{y}(t) = (c_1e^{-t}, c_2e^{-t})$.

## 25.7 Drawing Phase Portraits I

Suppose

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We could easily solve this equation, which has eigenvalues $\sigma(\mathbf{A}) = \{1, 3\}$. It's easy to see that $\lim_{t\to\infty} \|\mathbf{y}(t)\| = +\infty$ unless we start at the steady state, $(0, 0)$.

We will use the phase portrait to study the long-run behavior of this system. We see that $\dot{x} = 0$ when $2x + y = 0$ and $\dot{y} = 0$ when $x + 2y = 0$. These lines intersect at the steady state, $(0, 0)$. Above the $\dot{x} = 0$ line, $\dot{x}$ is positive. Below it, $\dot{x}$ is negative. Similarly, above the $\dot{y} = 0$ line, $\dot{y}$ is positive, and below it, $\dot{y}$ is negative.

Those lines are drawn on the figure below, breaking it into four "quadrants". The SE and NW "quadrants" are shaded. The arrows indicate the direction of motion within each of the four "quadrants".

In this case, the motion is away from the steady state at the origin, regardless of which "quadrant" you start in. The steady state is unstable. Starting near the steady state will lead to a trajectory that moves away from the origin.

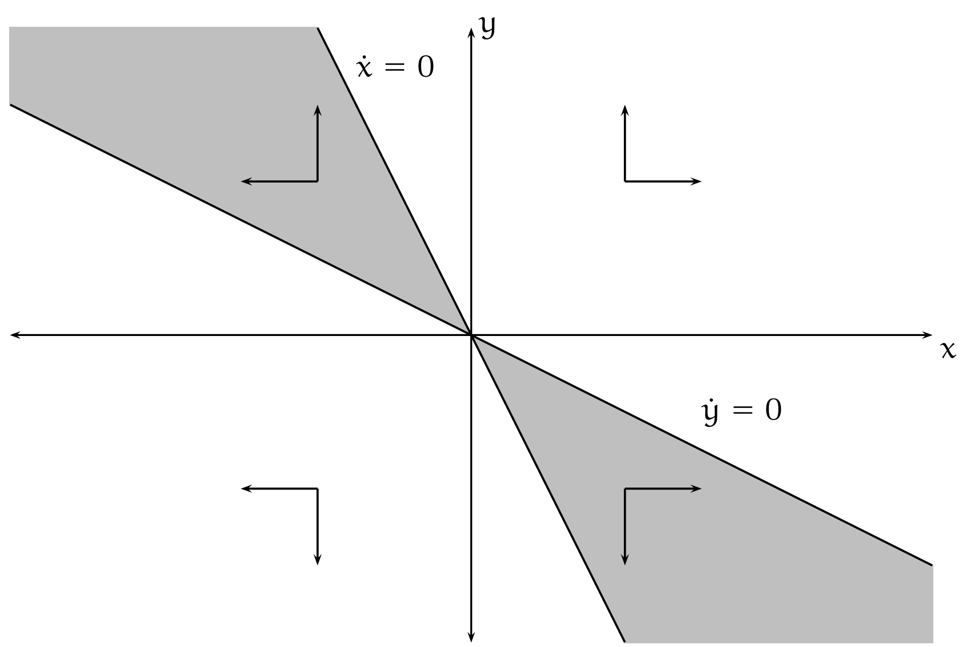


**Figure 25.7.1**: This diagram indicates the lines where $\dot{x} = 0$ and $\dot{y} = 0$ as when as the direction of motion (outward!) within each region.

## 25.8 Drawing Phase Portraits II

It's obvious that any trajectories starting in the NE or SW "quadrants" will move further NE or SW, off toward infinity. What happens in the gray zones is a little unclear, but you might imagine that one could start left of the $\dot{y} = 0$ axis in the SE and move toward it, before crossing it. The trajectory must be horizontal ($\dot{y} = 0$) at the crossing. It then moves off the the NE. Or perhaps one could start a little above the $\dot{x} = 0$ line, move SE to cross it vertically ($\dot{x} = 0$), then head SW.

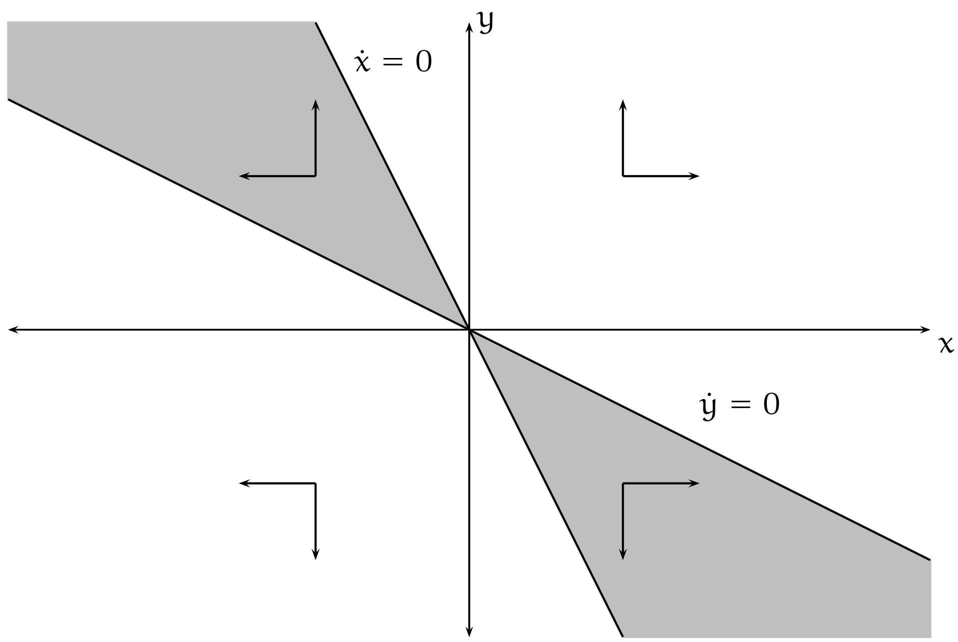


**Figure 25.7.1**: This diagram indicates the lines where $\dot{x} = 0$ and $\dot{y} = 0$ as when as the direction of motion (outward!) within each region.

## 25.9 Drawing Phase Portraits III

To sort the details out it helps to look at the eigenvectors. Here $\mathbf{v}_1 = (1,1)$ is an eigenvector for $\lambda = 3$ and $\mathbf{v}_2 = (1,-1)$ is an eigenvector for $\lambda = 1$. If we start on any multiple of the eigenvectors, we end up with solutions of the the form $c_1 e^{3t}(1,1)$ or $c_2 e^{t}(1,-1)$. Everywhere else, the trajectories are a blend of these types of solutions, meaning that the $e^{3t}$ term will eventually dominate.

For example, if we start in the SE cone at $(1,-0.8)$, then $c_1 + c_2 = 1$ and $c_1 - c_2 = -0.8$, implying $c_1 = 0.1$ and $c_2 = 0.9$. The trajectory starting at $(1,-0.8)$ is $\mathbf{y}(t) = (.1e^{3t} + 0.9e^{t}, .1e^{3t} - 0.9e^{t})$. This shows that $\lim_{t\to\infty} y_1/y_2 = 1$. The trajectory is asymptotic to a line parallel to the 45° line, even though it starts in the gray cone.

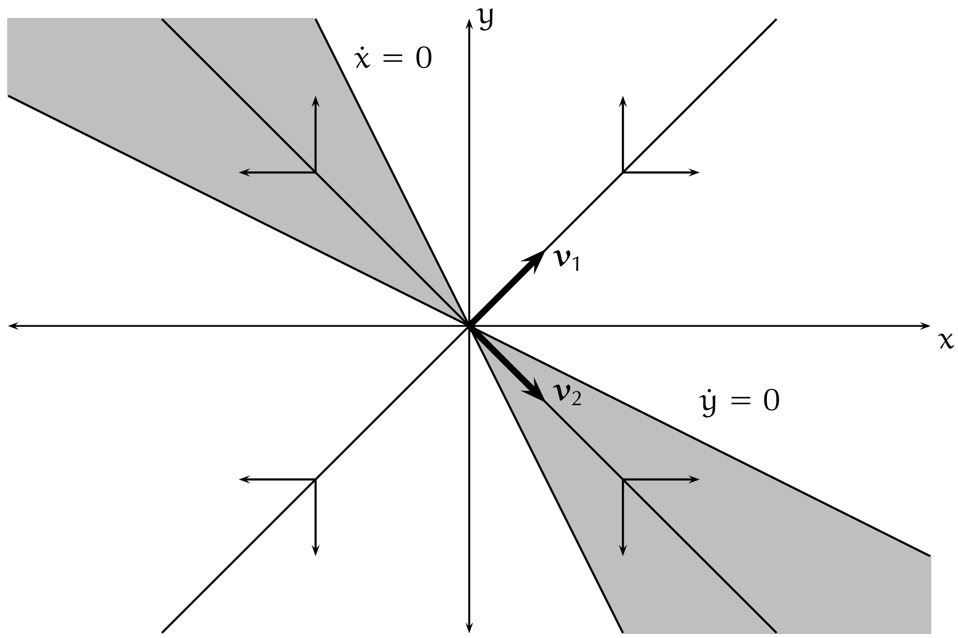


**Figure 25.9.1**: This diagram indicates the lines where $\dot{x} = 0$ and $\dot{y} = 0$ as when as the direction of motion (outward!) within each region. It also shows the eigenvectors (the heavy arrows) and their trajectories.

Keep in mind that the eigenvector lines cannot be crossed and that movement in the $(1,1)$ direction is faster than in the $(1,-1)$ direction.

## 25.10 Stability of Steady States

One question of interest is whether the steady states of a differential system are stable. A homogeneous system always has a steady state at $\mathbf{0}$. That is, $\mathbf{y}(t) = \mathbf{0}$ always solves the system

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{A}\mathbf{y} \\ \mathbf{y}(t_0) &= \mathbf{y}_0 \end{aligned} \tag{25.0.1}$$

when the initial data is $\mathbf{y}_0 = \mathbf{0}$.

For general autonomous systems $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y})$, a vector $\bar{\mathbf{y}}$ is a *steady state* if $\mathbf{F}(\bar{\mathbf{y}}) = \mathbf{0}$. It implies that $\mathbf{y}(t) = \bar{\mathbf{y}}$ solves the differential system with initial value $\bar{\mathbf{y}}$ because $\dot{\mathbf{y}} = \mathbf{F}(\bar{\mathbf{y}}) = \mathbf{0}$. For linear systems, the steady state condition becomes $\mathbf{A}\bar{\mathbf{y}} = \mathbf{0}$.

It follows that $\ker \mathbf{A}$ is the set of steady states for equation (25.0.1). The system has a unique steady state if $\mathbf{A}$ is invertible. If $\mathbf{A}$ is not invertible, any vector in $\ker \mathbf{A}$ is a steady state, meaning that the set of steady states is a vector subspace of $\mathbb{R}^m$.

Steady states need not be at zero for an inhomogeneous linear system. Let $\mathbf{y}_0$ be the initial data and

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{y} - \mathbf{y}_0)$$

Now $\dot{\mathbf{y}} = \mathbf{0}$ if $\mathbf{y} = \mathbf{y}_0$, so $\mathbf{y}_0$ is a steady state, and $\mathbf{y}(t) = \mathbf{y}_0$ solves the system.

We will have to make precise what we mean by "stable". There are many, many, different definitions in the literature. One author claimed there are over 100!. We will content ourselves with four.

The basic idea of stability is that trajectories starting near a steady state remain near it. In the case of asymptotic stability, those trajectories $\mathbf{y}(t)$ converge to that steady state.

## 25.11 Example: Purely Imaginary Roots

Before defining any type of stability, we first examine behavior near a steady state in $\mathbb{R}^2$. In these exercises, we will work in the eigenvector basis. If all of the eigenvalues are distinct, this is an orthonormal basis that is a rotation/reflection of the standard basis. We can also choose an orthonormal basis if the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity.

▶ Example 25.11.1: Circular Motion. Consider the system

$$\dot{y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y$$

The characteristic equation is $\lambda^2 + 1 = 0$, so the eigenvalues are $\sigma(\mathbf{A}) = \{+i, -i\}$. It follows that the general solution is

$$y = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}$$

Given initial data $y_0 = (x_0, y_0)$, we solve the problem by setting $c_1 = x_0$, $c_2 = -y_0$. Now

$$\begin{aligned} \|y(t)\|^2 &= x_0^2 \cos^2 t - 2x_0 y_0 \sin t \cos t + y_0^2 \sin^2 t \\ &\quad + y_0^2 \cos^2 t + 2x_0 y_0 \sin t \cos t + x_0^2 \sin^2 t \\ &= (x_0^2 + y_0^2)(\cos^2 t + \sin^2 t) \\ &= x_0^2 + y_0^2. \end{aligned}$$

This shows that the trajectory is a circle. If we start at a point at distance $\varepsilon$ from the steady state at $0$, we remain at distance $\varepsilon$. The trajectory does not approach the steady state, but doesn't move away either. This is illustrated in Figure 25.16.2.

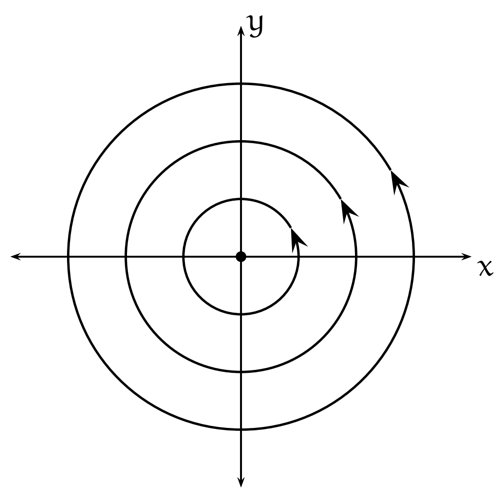


**Figure 25.11.2**: The phase portrait can be drawn either in $y$-space or in $(y_i, \dot{y}_i)$-space. Here $y_2 = \dot{y}_1$, so $y$-space is $(y_1, \dot{y}_1)$-space. Here we illustrate several trajectories. Each is an circle about the origin, which is our steady state. Each is also counter-clockwise about the steady state, as the arrows indicate.

◀

## 25.12 Example: Elliptical Motion

Elliptical motion can occur in second order linear systems in $\mathbb{R}^2$, which can be written as first order systems in $\mathbb{R}^4$.

▶ Example 25.12.1: Elliptical Motion. Consider the system $\ddot{\mathbf{y}} = -\mathbf{y}$ with initial data $\mathbf{y}(0) = (x_0, 0)$ and $\dot{\mathbf{y}}(0) = (0, y_0)$. It has solution

$$\mathbf{y}(t) = (x_0 \cos t)\boldsymbol{e}_1 + (y_0 \sin t)\boldsymbol{e}_2.$$

It follows that

$$\|\mathbf{y}(t)\|^2 = x_0^2 \cos^2 t + y_0^2 \sin^2 t$$

and the trajectory traces out an ellipse: $(y_1(t)/x_0)^2 + (y_2(t)/y_0)^2 = 1$.

This will be a circle if $x_0^2 = y_0^2$. The direction of motion is either clockwise or counter-clockwise, depending on the initial data. When $x_0 = y_0 = +1$ we get counter-clockwise motion, while if $x_0 = +1$, $y_0 = -1$, the motion is clockwise.

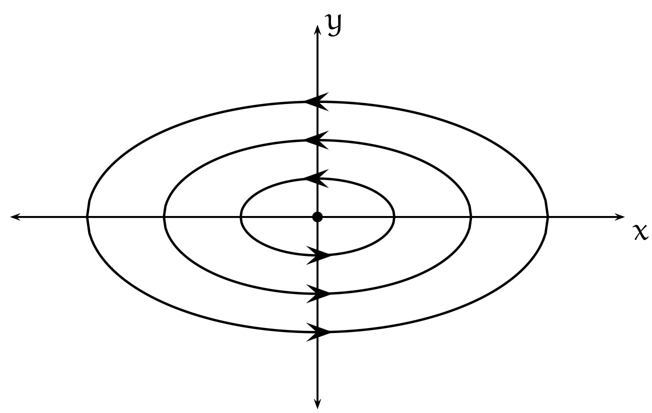


**Figure 25.12.2**: Here the phase portrait is drawn in $\mathbf{y}$-space. We illustrate three trajectories, with the direction of motion indicated. Each is an ellipse of the form $y_1^2 + 4y_2^2 = c^2$. The steady state is at $\mathbf{0}$.

◀

## 25.13 Lyapunov Stability

The circular and elliptical examples examined cases with a unique steady state. They have shown us that trajectories will sometimes orbit the steady state. Moreover those orbits can be eccentric—they can get nearer and farther from the steady state as they orbit around it.

Non-linear equations can lead to even more complex behavior. Lyapunov's notion of stability is intended to allow this.

**Lyapunov Stable.** The system (25.0.1) is *Lyapunov stable* at $\mathbf{0}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that $\|\mathbf{y}(t)\| < \varepsilon$ for all $t > t_0$ whenever $\|\mathbf{y}_0\| < \delta$.

Stability means that a trajectory $\mathbf{y}(t)$ stays within a distance $\varepsilon$ of the steady state if it starts within some distance $\delta$ of the steady state. The elliptical trajectories of Figure 25.12.2 are an example of Lyapunov stability. These trajectories are twice as far away at their farthest point than at their closest, so we would need to choose $\delta < \varepsilon/2$

## 25.14 Asymptotic Stability of Steady States

While Lyapunov stability only requires that trajectories remain near a steady state, the two types of asymptotic stability require convergence to a steady state.

**Locally Asymptotically Stable.** A steady state $\mathbf{0}$ is *locally asymptotically stable* if it is Lyapunov stable and there exists an $\varepsilon > 0$ so that $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ whenever $\|\mathbf{y}_0\| < \varepsilon$.

Local asymptotic stability takes Lyapunov stability and adds convergence to a steady state for trajectories that start nearby. Global asymptotic stability requires convergence to a steady state from any starting point. For this, the steady state must be unique.

**Globally Asymptotically Stable.** A steady state $\mathbf{0}$ is *globally asymptotically stable* if every solution to (25.0.1) converges to $\mathbf{0}$.

## 25.15 Stability in Linear Systems

In the case of a linear system, the stability is controlled by the eigenvalues.

***Theorem 25.15.1.*** *Consider a homogeneous linear first order differential system. If all of the eigenvalues* $\lambda \in \sigma(\mathbf{A})$ *satisfy* $\operatorname{Re}\lambda < 0$*, then the system is both globally and locally asymptotically stable.*

**Proof.** We know the solution has the form

$$\mathbf{y}(t) = e^{(t-t_0)\mathbf{A}}\mathbf{y}_0.$$

We can write it as

$$\mathbf{y}(t) = \mathbf{P}e^{(t-t_0)\mathbf{J}}\mathbf{P}^{-1}\mathbf{y}_0$$

where $\mathbf{J}$ is in Jordan canonical form.

Now

$$\|\mathbf{y}(t)\| \leq \|\mathbf{P}\|\|e^{(t-t_0)\mathbf{J}}\|\|\mathbf{P}^{-1}\|\|\mathbf{y}_0\|$$

where we use the $\ell^\infty$ matrix norm. Now each term of $e^{(t-t_0)\mathbf{J}}$ is a most a constant times a polynomial in $t$ and the exponential of $\operatorname{Re}\lambda t$. The last goes to zero fast enough to swamp the polynomial, so all terms (and the norm) converge to zero. Since this is bounded, we have Lyapunov stability. Since it converges to zero, we get both local and global asymptotic stability. ∎

## 25.16 Saddlepoint Stability

There is one other type of stability that we will be interested in, *saddlepoint stability*. With saddlepoint stability, there is a manifold (vector subspace for linear equations) so that trajectories starting on the manifold converge to the steady state, and trajectories starting elsewhere do not converge to the steady state.

An example will help clarify how this works.

► Example 25.16.1: Saddlepoint Stability. Let

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenvalues are $\sigma(\mathbf{A}) = \{2, -1\}$ and the general solution is $\mathbf{y}(t) = c_1 e^{2t}\mathbf{e}_1 + c_2 e^{-t}\mathbf{e}_2$.

If we start on the vertical axis, with $\mathbf{y}_0 = \alpha\mathbf{e}_2$, the coefficients are $c_1 = 0$ and $c_2 = \alpha$. That yields solution $\mathbf{y}(t) = \alpha e^{-t}\mathbf{e}_2$. It follows that $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$, showing the steady state is stable in this direction.

But it is unstable in any other direction. If $\mathbf{y}_0$ has any other form, $c_1 \neq 0$, and $\lim_{t\to\infty} \mathbf{y}(t) = (\operatorname{sgn} c_1)\infty$. This system is saddlepoint stable.

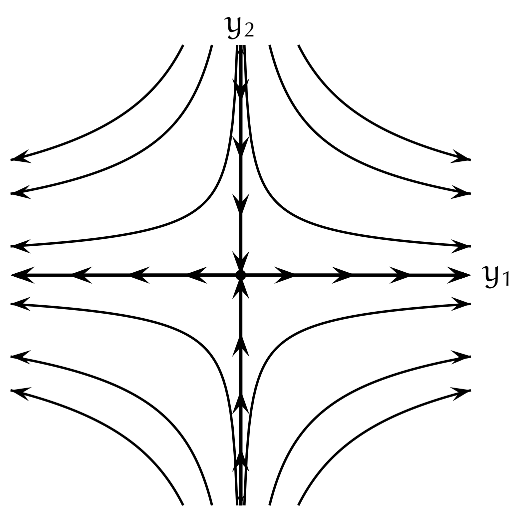


**Figure 25.16.2**: We illustrate several trajectories, with the arrows indicating the direction of motion. Notice how solutions starting on the vertical axis converge to the steady state at the origin, while trajectories starting anywhere else are asymptotic to the horizontal axis.

◄

## 25.17 The $m = 2$ Case

When studying stability, it will be useful to examine the $m = 2$ case in detail. The intuition gained here will tell use much about how stability works for arbitrary $m$.

Consider the system

$$\dot{y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} y$$

where $a$, $b$, $c$, and $d$ are real numbers.

The characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. This can be written in terms of the trace, $\operatorname{tr} \mathbf{A} = (a + d)$ and determinant $\det \mathbf{A} = (ad - bc)$ as

$$\lambda^2 - (\operatorname{tr} \mathbf{A})\lambda + \det \mathbf{A} = 0.$$

This has solutions

$$\lambda_i = \frac{(\operatorname{tr} \mathbf{A}) \pm \sqrt{(\operatorname{tr} \mathbf{A})^2 - 4(\det \mathbf{A})}}{2}$$

for $i = 1, 2$.

The expression

$$\Delta = (\operatorname{tr} \mathbf{A})^2 - 4(\det \mathbf{A})$$

is called the *discriminant*. It tells us whether the roots are complex (if and only if $\Delta < 0$), identical ($\Delta = 0$), or real and distinct ($\Delta > 0$).

We found earlier that $\operatorname{tr} \mathbf{A} = \lambda_1 + \lambda_2$ and $\det \mathbf{A} = \lambda_1\lambda_2$ where the $\lambda_i$ are the two eigenvalues of $\mathbf{A}$. We can use this fact to characterize the types of solutions in terms of the trace and determinant.

Applied to the discriminant, we obtain

$$\Delta = (\operatorname{tr} \mathbf{A})^2 - 4(\det \mathbf{A}) = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2.$$

The discriminant is zero if and only if $\lambda_1 = \lambda_2$. Provided that any complex eigenvalues are conjugates, the discriminant is positive if and only if the eigenvalues are real and distinct, and negative if and only if they are complex conjugates.

## 25.18 Determinant, Discriminant, and Trace

We can now use the trace ($\operatorname{tr} \boldsymbol{A} = \lambda_1 + \lambda_2$), determinant ($\det \boldsymbol{A}$) and discriminant ($(\lambda_1 - \lambda_2)^2$) to classify the types of solutions to 2-dimensional homogeneous first order linear systems with real coefficients.

The sign of the discriminant determines whether the roots are real and distinct ($\Delta > 0$), real and identical ($\Delta = 0$) or complex conjugates ($\Delta < 0$).

The sign of the determinant is the sign of $\lambda_1\lambda_2$. It is positive if the roots are either (1) complex, (2) both positive, or (3) both negative. It is zero if one or both roots are zero. It is negative if the roots are real with opposite signs.

As for the trace, its sign is the sign of the real part of the roots when they are complex conjugates, its sign is the sign of the sum when both roots are real.

In the end, there are six regions where all three are not zero. We also have to consider the borderline cases where one or more is zero.

## 25.19 Types of solutions in $\mathbb{R}^2$

The various possibilities are shown in Figure 25.19.1.[1] The $\Delta = 0$ curve in the figure is defined by $(\operatorname{tr}\boldsymbol{A})^2 = 4\det\boldsymbol{A}$. Above the curve the roots are complex conjugates, on it they are real and equal, and below it both roots are real and distinct. Below the $\operatorname{tr}\boldsymbol{A}$ axis, where $\det\boldsymbol{A} = 0$, one root is negative and one is positive. Above it, the roots are either complex, or real with with same sign. For the real roots with identical signs, the sign is determined by the trace, which is positive when both are positive, negative when both are negative. For complex conjugate roots, the sign of the trace is the sign of the real part.

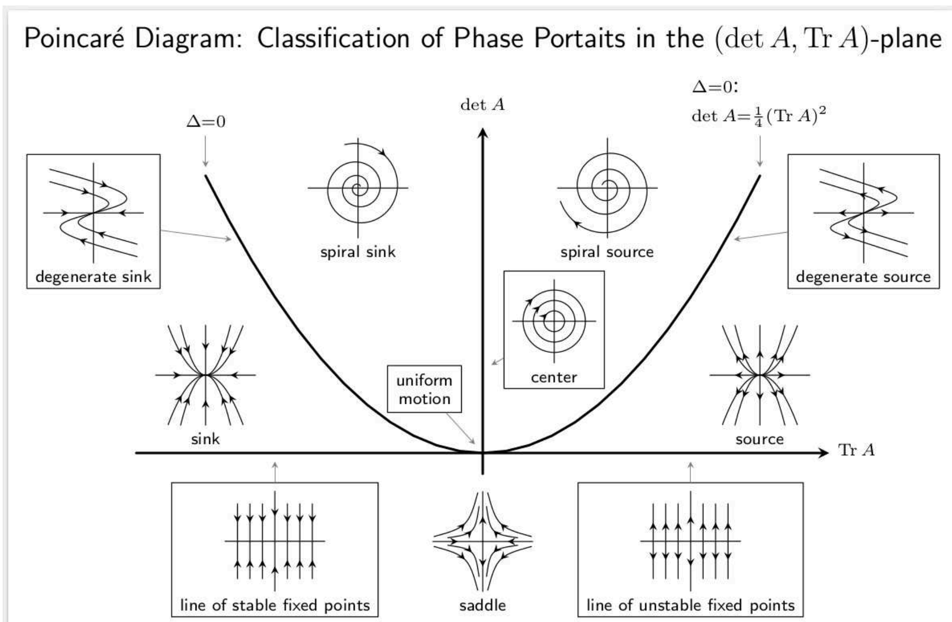


**Figure 25.19.1**: This figure shows the various possible phase portraits for a two-dimensional first order linear differential system $\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}$ The system has characteristic polynomial $\lambda^2 - (\operatorname{tr}\boldsymbol{A})\lambda + \det\boldsymbol{A}$. The diagram indicates the phase portraits for various values of the parameters $\operatorname{tr}\boldsymbol{A}$ and $\det\boldsymbol{A}$, which are plotted on the horizontal and vertical axes, respectively.

[1] The diagram was created by the Wikipedia user Freesodas, and is licensed under the Creative Commons Attribution-ShareAlike 4.0 International license:
https://creativecommons.org/licenses/by-sa/4.0/deed.en.

## 25.20 Stability in $\mathbb{R}^2$: Summing Up

Figure 25.19.1 illustrated the possibilities.

We can characterize the behavior away from the steady state as follows:

(a) Spiral Source or Sink: If the roots are real, both are positive for a source, negative for a sink. If the roots are complex conjugates, a positive real part indicates a source, a negative real part means it is a sink. Finally, if the roots are identical (and necessarily real), the source or sink is degenerate.

(b) Circles: These only occur when the roots are purely imaginary, meaning $\operatorname{tr} \mathbf{A} = 0$. The direction of the circular motion depends on the initial data.

(c) Saddlepoint Stability: This happens if one root is positive and one root is negative ($\det \mathbf{A} < 0$). The trace determines whether the positive root is larger or smaller than the negative root, but doesn't affect the form of the trajectories.

(d) Line of Steady States: If $\det \mathbf{A} = 0$, one of the roots is zero. If also $\operatorname{tr} \mathbf{A} = 0$, both roots are zero and every point is a steady state. If instead $\operatorname{tr} \mathbf{A} \neq 0$, we get a line ($\ker \mathbf{A}$) of steady states. These will be stable if $\operatorname{tr} \mathbf{A} < 0$, when the non-zero eigenvalue is negative. They will be unstable if $\operatorname{tr} \mathbf{A} > 0$, when the non-zero eigenvalue is positive.

The saddlepoint case is of particular importance in many economic models, where the only allowable solutions will be on the stable arm. This fact will let us get by with a single initial condition. The second is replaced by another condition that forces the solution to be on the stable arm.

# 34. Mathematics of Optimal Growth

**NB:** The Final is on Thursday, December 10 from 5-7 pm. **12/3/20**

There are often advantages to writing optimal growth models in continuous time rather than discrete time. Continuous time allows for a sharp distinction between stocks and flows. It is also sometimes easier to analyze the behavior of the problem near a steady state. In some case, chaotic behavior is ruled out in continuous time due to the inability of continuous variable to jump over intermediate points.

## 34.1 An Optimal Growth Problem

It is sometimes convenient to write optimal growth models in continuous time rather than discrete time. We will briefly examine such a model. A capital stock $k(t)$ results in an output flow of $f\big(k(t)\big)$, where $f$ is the production function. The output flow is divided into a consumption flow $c(t)$ and an investment flow $dk/dt$. Thus

$$f\big(k(t)\big) = c(t) + \dot{k}(t)$$

The consumer receives utility at rate $u(c)$ from the consumption flow $c$. This is discounted at rate $r$, so the utility flow at time $t$ is $e^{-rt}u\big(c(t)\big)$ in present value terms. We integrate this over time to find total utility of the consumption path $c$. It is

$$U(c) = \int_0^\infty e^{-rt}u\big(c(t)\big)\,dt.$$

The consumer starts with an initial capital stock $k(0) = k_0$. The consumer then maximizes utility $U(c)$ over all consumption paths $c$ that obey $c + \dot{k} = f(k)$, $k(0) = k_0$.

The calculus of variations can be used to solve such problems.[1] Before applying it, we substitute $c = f(k) - \dot{k}$. We seek to maximize

$$\int_0^\infty e^{-rt}u\big(f(k) - \dot{k}\big)\,dt$$

subject to the initial condition $k(0) = k_0$.

[1] The problem of maximizing or minimizing functionals, mappings from a space of functions to the real numbers, occurred early in the history of calculus. Johann Bernoulli solved a brachistochrone problem in 1696. The term "calculus of variations" is due to Leonhard Euler. He and Joseph-Louis Lagrange were responsible for much of the development of the general theory. Modern developments include optimal control theory and dynamic programming.

## 34.2 The Calculus of Variations

Whereas the ordinary calculus takes derivatives by looking at small variations in a single variable about a point $x_0$, the calculus of variations varies the function $k$ by adding another (small) function. To do this, take an arbitrary infinitely differentiable function $\varphi(t)$ such that $\varphi(0) = 0$ and $\varphi(t) = 0$ for $t$ sufficiently large.[2] We form

$$\Phi(\varepsilon) = \int_0^\infty e^{-rt} u\big(f(k + \varepsilon\varphi) - \dot{k} - \varepsilon\dot{\varphi}\big)\, dt.$$

We maximize by setting $\Phi'(0) = 0$, obtaining

$$0 = \int_0^\infty e^{-rt} u'\big(c(t)\big)\big[f'\big(k(t)\big)\varphi(t) - \dot{\varphi}(t)\big]\, dt.$$

We integrate the second term by parts. The fact that $\varphi(0) = 0$ and $\varphi(t) = 0$ for $t$ sufficiently large implies

$$\begin{aligned}\int_0^\infty & e^{-rt} u'(c)\dot{\varphi}\, dt \\ &= e^{-rt} u'(c)\,\varphi(t)\Big|_0^\infty - \int_0^\infty \frac{d}{dt}\big[e^{-rt}u'(c)\big]\,\varphi\, dt \\ &= -\int_0^\infty \frac{d}{dt}\big[e^{-rt}u'(c)\big]\,\varphi\, dt.\end{aligned}$$

Thus

$$0 = \int_0^\infty \left[e^{-rt}u'\big(c(t)\big)f'\big(k(t)\big) + \frac{d}{dt}\Big[e^{-rt}u'\big(c(t)\big)\Big]\right]\varphi(t)\, dt. \tag{34.2.1}$$

Now we invoke the following theorem from section 32.4 of my manuscript.

***Theorem 32.4.7.*** *Let $I$ be an open interval in $\mathbb{R}$. Suppose $f\colon I \to \mathbb{R}$ is an integrable function. If $\int_I f(x)\varphi(x)\, dx = 0$ for all $\varphi \in \mathcal{C}_c^\infty(I)$ or $\varphi \in \mathcal{C}_c(I)$, then $f = 0$, a.e. Moreover, if $f$ is also continuous, $f = 0$ on $I$.*

[2] These are functions with compact support, meaning that $\varphi(t) = 0$ for $t$ outside some compact set, in this case outside some interval $[0, T]$.

## 34.3 The Euler-Lagrange Equations

Since equation (34.2.1) holds for all $\varphi \in \mathcal{C}^\infty$ with compact support, we can conclude

$$e^{-rt}u'\big(c(t)\big)f'\big(k(t)\big) + \frac{d}{dt}\Big[e^{-rt}u'\big(c(t)\big)\Big] = 0.$$

which is the first order condition for this problem. We rewrite this as

$$e^{-rt}u'\big(c(t)\big)f'\big(k(t)\big) = -\frac{d}{dt}\Big[e^{-rt}u'\big(c(t)\big)\Big].$$

More generally, if $u$ is a function of $k$ and $\dot{k}$, we find the *Euler-Lagrange equation*,

$$e^{-rt}\frac{\partial u}{\partial k} = \frac{d}{dt}\left[e^{-rt}\frac{\partial u}{\partial \dot{k}}\right].$$

The sign flip is due to the fact that $c = f(k) - \dot{k}$, so $\partial c/\partial \dot{k} = -1$.

Usually, the Euler-Lagrange equation will have many solutions, and most of them will not solve the original optimization problem. One more condition is needed , the transversality condition. The transversality condition for problems of this sort is

$$\lim_{t\to\infty} e^{-rt}u'\big(c(t)\big)\,k(t) = 0.$$

We can use the Euler-Lagrange equation and transversality condition to solve optimal growth problems in continuous time.

## 34.4 Continuous-time Optimal Growth I

Let $u(c) = c^{1-\sigma}/(1-\sigma)$ for $\sigma > 0$, $\sigma \neq 1$ and $f(k) = \alpha k$. Then the Euler-Lagrange equation is

$$\begin{aligned} e^{-rt}\frac{\partial u}{\partial k} &= \frac{d}{dt}\left[e^{-rt}\frac{\partial u}{\partial \dot{k}}\right]. \\ \alpha e^{-rt}c^{-\sigma} &= -\frac{1}{1-\sigma}\frac{d}{dt}\left[e^{-rt}\frac{\partial c^{1-\sigma}}{\partial c}\right] \\ &= re^{-rt}c^{-\sigma} + \sigma e^{-rt}c^{-\sigma-1}\dot{c}. \end{aligned}$$

We can clear the $e^{-rt}$ term, yielding

$$(\alpha - r)c^{-\sigma} = \sigma c^{-\sigma-1}\dot{c},$$

which simplifies to

$$(\alpha - r)/\sigma = \frac{\dot{c}}{c}.$$

This can be integrated to obtain $c(t) = c_0 e^{(\alpha-r)t/\sigma}$ for some constant $c_0$.

It immediately follows that consumption increases over time when $\alpha > r$ and decreases when $\alpha < r$. That is, consumption and capital stock increase when the rate of return $\alpha$ is larger than the interest rate $r$, and decrease if it is smaller.

To finish solving the problem, we must determine $c_0$. Total utility $U(c)$ will only make sense if $[\alpha(1-\sigma) - r] < 0$, which we will assume for the remainder of the problem.[3]

We next solve for $k(t)$. We know

$$\dot{k} = f(k) - c = \alpha k - c_0 e^{(\frac{\alpha - r}{\sigma})t}.$$

We solve this by grouping the $k$ terms together and multiplying by the integrating factor $e^{-\alpha t}$, obtaining

$$e^{-\alpha t}\left[\dot{k} - \alpha k\right] = -c_0 e^{(\alpha - \sigma\alpha - r)t/\sigma}.$$

The left-hand side is $d[e^{-\alpha t}k]/dt$, which allows us to integrate both sides, obtaining

$$e^{-\alpha t}k(t) - k_0 = -\frac{c_0\sigma}{\alpha(1-\sigma) - r}\left[e^{\frac{\alpha(1-\sigma)-r}{\sigma}t} - 1\right].$$

This can be written

$$k(t) = \left[k_0 + \frac{c_0\sigma}{\alpha(1-\sigma) - r}\right]e^{\alpha t} - \frac{c_0\sigma}{\alpha(1-\sigma) - r}e^{\frac{\alpha - r}{\sigma}t}.$$

[3] Although we assumed $\sigma \neq 1$, the same arguments steps apply when $u(c) = \ln c$, meaning $\sigma = 1$. In that case utility converges regardless of $\alpha$ and $r$.

## 34.5 Continuous-time Optimal Growth II

We now employ the transversality condition to pin down $c_0$.

$$\lim_{t\to\infty} p(t)k(t) = 0.$$

Here $p(t)$, the implied price, is the discounted flow of marginal utility,

$$p = e^{-rt}u'(c) = e^{-rt}c^{-\sigma} = c_0^{-\sigma}e^{-\alpha t}.$$

Then the transversality condition is
Now

$$p(t)k(t) = c_0^{-\sigma}\left[k_0 + \frac{c_0\sigma}{\alpha(1-\sigma)-r}\right] - \frac{c_0\sigma}{\alpha(1-\sigma)-r}e^{\frac{\alpha(1-\sigma)-r}{\sigma}t}$$

We are already assuming that $\alpha(1-\sigma)-r<0$, so the second term converges to zero. The transversality condition becomes

$$0 = \lim_{t\to\infty} p(t)k(t) = c_0^{-\sigma}\left[k_0 + \frac{c_0\sigma}{\alpha(1-\sigma)-r}\right].$$

which requires

$$k_0 = -\frac{c_0\sigma}{\alpha(1-\sigma)-r} \quad \text{meaing} \quad c_0 = -\left(\frac{\alpha(1-\sigma)-r}{\sigma}\right)k_0.$$

It follows that

$$k(t) = k_0 \exp\left(\frac{\alpha-r}{\sigma}\right)t$$

and

$$c(t) = k_0\left(\frac{r-\alpha(1-\sigma)}{\sigma}\right)\exp\left(\frac{\alpha-r}{\sigma}\right)t.$$

Note that consumption and capital either both grow or shrink at the same rate, $(\alpha-r)/\sigma$.

## 34.6 Optimal Growth with Logarithmic Utility

Consider the continuous-time optimal growth problem for $u(c) = \log c$ and $f(k) = \alpha k$ with $2r > \alpha > r$.

The Euler-Lagrange equation is

$$\frac{\alpha e^{-rt}}{\alpha k - \dot{k}} = -\frac{d}{dt}\left[\frac{e^{-rt}}{\alpha k - \dot{k}}\right] = r\frac{e^{-rt}}{\alpha k - \dot{k}} + \frac{e^{-rt}(\alpha\dot{k} - \ddot{k})}{(\alpha k - \dot{k})^2}.$$

This simplifies to

$$(\alpha - r)(\alpha k - \dot{k}) = \frac{d}{dt}(\alpha k - \dot{k}).$$

Let $z = \alpha k - \dot{k}$. The equation becomes $(\alpha - r)z = \dot{z}$ with general solution

$$\alpha k(t) - \dot{k}(t) = z(t) = Ae^{(\alpha - r)t}.$$

Multiply both sides by the integrating factor $e^{-\alpha t}$, obtaining

$$(d/dt)(e^{-\alpha t}k) = -Ae^{-rt}.$$

Integrating, we find $e^{-\alpha t}k(t) = B + Ce^{-rt}$ for some constants B and C. Thus

$$k(t) = Be^{\alpha t} + Ce^{(\alpha - r)t}.$$

We know $k(0) = k_0$, so $B + C = k_0$.

We now appeal to the transversality condition,

$$\lim_{t\to\infty} e^{-rt}u'\big(c(t)\big)\, k(t) = 0.$$

Since $\alpha > r$, this requires $B = 0$, so $C = k_0$ and $k(t) = k_0 e^{(\alpha - r)t}$. Note that $2r > \alpha$ ensures that the transversality condition is satisfied.

## 34.7 Continuous Time Consumer's Problem I

We can use a similar technique when the consumer maximizes $\int_0^\infty u(c)e^{-rt}\,dt$ under the constraint that the discounted value of consumption is $W$. I.e., when $\int_0^\infty c(t)e^{-rt}\,dt = W$.

As before, consider

$$\Phi(\varepsilon) = \int_0^\infty u(c + \varepsilon\varphi)e^{-rt}\,dt, \quad \text{s.t.} \quad \int_0^\infty (c + \varepsilon\varphi)e^{-rt}\,dt = W.$$

We find

$$\int_0^\infty u'(c)e^{-rt}\varphi\,dt = 0 \tag{34.7.2}$$

for all $\varphi$ such that

$$\int_0^\infty \varphi(t)e^{-rt}\,dt = 0.$$

Unfortunately, the restriction on $\varphi$ means that Theorem 32.4.7 doesn't directly apply. Our integral equation does not hold for all $\mathcal{C}^\infty$ functions with compact support. Rather, it holds for a set of $\varphi \in \mathcal{C}^\infty$ with co-dimension 1.

## 34.8 Continuous Time Consumer's Problem II

The solution is to write an arbitrary $\mathcal{C}^\infty$ function $\psi$ as a linear combination of a function obeying the constraint, and a standard function that does not obey the constraint. Since we are integrating the function against $e^{-rt}$, we can use a constant term.

Then $\int_0^\infty e^{-rt}\,dt = 1/r$, which does not obey the constraint. This allows us to write

$$\varphi(t) = \psi(t) - r\left(\int_0^\infty \psi(t)e^{-rt}\,dt\right)$$

where $\varphi$ obeys $\int_0^\infty \varphi(t)e^{-rt}\,dt = 0$. As for $\psi$, we only require that it be a $\mathcal{C}^\infty$ function of compact support. The function $\psi$ will take the place of $\varphi$.

$$\begin{aligned}\int_0^\infty u'(c)\psi e^{-rt}\,dt = \\ &\int_0^\infty u'(c)\varphi e^{-rt}\,dt + r\left(\int_0^\infty u'(c)e^{-rs}\,ds\right)\times\left(\int_0^\infty \psi e^{-rt}\,dt\right).\end{aligned}$$

The first term on the right-hand side is zero by equation (34.7.2). This lets us rewrite the equality as

$$\int_0^\infty \left(u'(c)e^{-rt} - re^{-rt}\left(\int_0^\infty u'(c)e^{-rs}\,ds\right)\right)\psi\,dt = 0.$$

As this holds for all $\psi \in \mathcal{C}^\infty$ with compact support, we can now appeal to Theorem 32.4.7. This yields

$$u'\big(c(t)\big)e^{-rt} = re^{-rt}\int_0^\infty u'(c)e^{-rs}\,ds$$

so

$$u'\big(c(t)\big) = r\int_0^\infty u'(c)e^{-rs}\,ds.$$

The right-hand side is independent of $t$, implying that $u'\big(c(t)\big)$ is constant. It follows that $c(t) = c_0$ for all $t$. Finally, we use the constraint $\int_0^\infty c(t)e^{-rt}\,dt = W$ to find that $c_0 = rW$.

## 34.9 The Consumer's Problem with Logarithmic Utility

We apply this to the problem

$$\max_c \int_0^\infty e^{-rt} \ln c \, dt$$
$$\text{s.t.} \int_0^\infty e^{-zt} c(t)\, dt = W.$$

Here the discount rates differ for consumption and utility, unlike the example above.

This is a variation on the intertemporal consumer's problem. We again define

$$\Phi(\varepsilon) = \int_0^\infty u(c + \varepsilon\varphi)e^{-rt}\, dt$$

where $\varphi$ obeys $0 = \int_0^\infty e^{-zt}\varphi(t)\, dt$ (the latter condition is to maintain feasibility of $c + \varepsilon\varphi$).

Set $\Phi'(0) = 0$, we obtain $\int u'(c)e^{-rt}\varphi(t)\, dt = 0$. This holds for any $\varphi$ with $0 = \int_0^\infty e^{-zt}\varphi(t)\, dt$. We use the same trick as before, writing $\varphi(t) = \psi(t) - z\big(\int_0^\infty \psi(t)e^{-zt}\, dt\big)$. Then $\int e^{-zt}\varphi(t)\, dt = 0$, so

$$\int_0^\infty u'(c)e^{-rt}\varphi\, dt = \int_0^\infty u'(c)e^{-rt}\psi\, dt - z\left(\int_0^\infty u'(c)e^{-rt}\, dt\right)\left(\int_0^\infty e^{-zt}\psi\, dt\right)$$

The left-hand side is zero, and we can rewrite the right-hand side as before, obtaining

$$0 = \int_0^\infty \left(u'(c)e^{-rt} - ze^{-zt}\left(\int_0^\infty u'(c)e^{-rs}\, ds\right)\right)\psi\, dt.$$

Using Theorem 32.4.7, we conclude

$$e^{-rt}u'(c) = ze^{-zt}\int_0^\infty u'(c)e^{-rs}\, ds.$$

This means that $u'\big(c(t)\big) = 1/c(t)$ is a constant times $e^{(r-z)t}$, or $c(t) = Ae^{(z-r)t}$ for some $A$. We now employ the budget constraint, $\int Ae^{(z-r)t}e^{-zt}\, dt = W$. In other words, $(A/r) = W$, or $A = rW$. The solution is

$$c(t) = rWe^{(z-r)t}. \tag{34.9.3}$$

Essentially, this is a consumer's problem with infinite-horizon Cobb-Douglas utility. The raw weights are $e^{-rt}$. These integrate to $1/r$, giving normalized weights of $re^{-rt}$. Finally, the price of consumption at time $t$ is $e^{-zt}$, so we divide by that, obtaining equation (34.9.3).

## 34.10 Optimal Control Theory

In optimal control theory, variables are divided into two classes—*control variables*, which we control, and *state variables*, which respond to the controls. A typical problem in optimal control theory is to pick a function $c$ which maximizes the objective

$$U(c) = \int_0^\infty u(t, k, c)\, dt$$

with $k$ obeying the differential equation $\dot{k} = g(t, k, c)$ and initial conditions $k(0) = k_0$, and $c$ subject to a possibly time-varying constraints, $c(t) \in A(t)$. Here $k$ is the state variable and $c$ is the control variable. We write $U(c)$ to emphasize the dependence of utility on the control $c$. The problem is often written as follows:

$$\begin{aligned} \max_c \int_0^\infty & u(t, k, c)\, dt \\ \text{s.t. } \dot{k} &= g(t, k, c) \\ c &\in A(t) \\ k(0) &= k_0 \end{aligned}$$

## 34.11 Costate Variables

The key step in solving this is to introduce an auxiliary variable $p$, which is in some sense dual to the state. Let $p$ be an arbitrary differentiable function. Given the constraints, the objective can be rewritten

$$\int_0^T p_0 u(t,k,c) + pg(t,k,c) - p\dot{k}\,dt.$$

At this point we have placed no conditions on $p$, thus we may assume the transversality condition that $p(T) = 0$ if $T$ is finite or if $T = +\infty$, that $p(t)k(t) \to 0$ as $t \to \infty$. Integrating the last term by parts yields

$$U(k) = p(0)k(0) + \int_0^T p_0 u(t,k,c) + pg(t,k,c) + \dot{p}k\,dt.$$

We again form $\Phi(\varepsilon)$ by altering $c$ in the direction $\varphi$ with $c + \varepsilon\varphi \in A(t)$ for small $\varepsilon$. Let $k(t|\varepsilon)$ denote the solution to $\dot{k} = g(t,k,c+\varepsilon\varphi)$, $k(0) = k_0$. When $c$ is continuous and $g$ is continuously differentiable, the Picard-Lindelöf Theorem guarantees that a unique $k(t|\varepsilon)$ exists,[4] and is differentiable with respect to $\varepsilon$.[5] Thus $k(t) = k(t|0)$. This yields

$$\Phi(\varepsilon) = p(0)k(0) + \int_0^T p_0 u\big(t, k(t|\varepsilon), c+\varepsilon\varphi\big) + pg\big(t, k(t|\varepsilon), c+\varepsilon\varphi\big) + \dot{p}k(t|\varepsilon)\,dt.$$

We now take $\Phi'$.

$$\Phi'(\varepsilon) = \int_0^T (p_0 u_c + pg_c)\varphi + (p_0 u_k + pg_k + \dot{p})\frac{dk(t|\varepsilon)}{dt}\,dt$$

This holds for any $p$, in particular, when $p$ solves the differential equation $-\dot{p} = p_0 u_k + pg_k$. Such a $p$ is called the *costate variable*. Using the costate means that the second term is zero, so

$$\Phi'(\varepsilon) = \int_0^T (p_0 u_c + pg_c)\varphi\,dt$$

Since $\Phi$ is maximized at $\varepsilon = 0$, $\Phi'(0) = 0$ and the equation simplifies to

$$0 = \int_0^T (p_0 u_c + pg_c)\varphi\,dt$$

with $k(t)$ and $c(t)$ in the functions $u_c$ and $g_c$.

If $A(t) = \mathbb{R}$, we have $p_0 u_c + pg_c = 0$. Otherwise, $(p_0 u_c + pg_c)\varphi \le 0$ for all $\varphi$ with $c + \varepsilon\varphi \in A(t)$ for $\varepsilon$ small. When $u$ and $g$ are concave, either case means we are at the maximum of $p_0 u + pg$.

[4] Hartman, 1982, pg. 8; Coddington and Levinson, 1955, pg. 12

[5] Hartman pg. 95; Coddington and Levinson, pg. 25

## 34.12 The Maximum Principle

What we've derived is a version of the Maximum Principle, due to Pontryagin and his students in the 1950's. More generally, define the *Hamiltonian* by

$$\mathcal{H}(t, k, c, p) = p_0 u(t, k, c) + p(t) g(t, k, c).$$

Note that $\mathcal{H}_k = p_0 u_k + p g_k$ and $\mathcal{H}_p = g$.

***Maximum Principle.*** *Suppose $u$ and $g$ are concave and continuously differentiable. Then differentiable $k^*$ and continuous $c^*$ solve*

$$\begin{aligned} \max_c \int_0^\infty & u(t, k, c)\, dt \\ \text{s.t. } \dot{k} &= g(t, k, c) \\ c &\in A(t) \\ k(0) &= k_0 \end{aligned}$$

*if and only if there is a differentiable function $p(t)$ and constant $p_0 \geq 0$ such that:*

(1) $\dot{p} = -\mathcal{H}_k$ *(costate equation)*
(2) $\dot{k} = \mathcal{H}_p$ *(state equation)*
(3) $\mathcal{H}(t, k^*, c^*, p) \geq \mathcal{H}(t, k^*, c, p)$ *for all* $c \in A(t)$
(4) $\lim_{t\to\infty} p(t)k(t) = 0$ *(transversality condition)*
(5) $k^*(0) = k_0$ *(initial condition).*

*where* $\mathcal{H} = u(t, k, c) + p(t)g(t, k, c)$.

Notice that the number of state variables need not equal the number of controls. Just like in the Kuhn-Tucker Theorem, the constant $p_0$ is a substitute for certain constraint qualification conditions that we have glossed over. In most economic applications, $p_0$ will be non-zero. We can then assume $p_0 = 1$ without loss of generality.

## 34.13 Continuous Time Consumer's Problem Revisited

The consumer faces an interest rate $r > 0$. The discounted value of consumption up to time $t$ is $\int_0^t e^{-rs} c(s)\, ds$. The amount of wealth remaining at time $t$ is then $k(t) = W - \int_0^t e^{-rs} c(s)\, ds$. This will be our state variable. Then we have $\dot{k} = -e^{-rt} c(t)$ and $k(0) = W$. The consumer discounts utility $u(c) = \ln c$ at the rate of impatience $\rho > 0$. The intertemporal consumer's problem can then be written as:

$$\begin{aligned} \max_c \; & \int_0^\infty e^{-\rho t} \ln\big(c(t)\big)\, dt \\ \text{s.t. } & \dot{k} = -e^{-rt} c(t) \\ & c \geq 0 \\ & k(0) = W. \end{aligned}$$

We form the Hamiltonian

$$\mathcal{H} = e^{-\rho t} u\big(c(t)\big) - e^{-rt} p(t) c(t).$$

By the Maximum Principle, the costate equation is $\dot{p} = -\mathcal{H}_k = 0$. Solving, we find $p(t) = p(0)$ is constant. The state equation is $\dot{k} = \mathcal{H}_p = -e^{-rt} c(t)$.

Next we must maximize the Hamiltonian. Obviously, the constraint $c \geq 0$ can never bind with this utility function, so we maximize the Hamilton by setting the $c$ derivative to zero, $\mathcal{H}_c = 0$. Thus $e^{-\rho t}/c = p(0) e^{-rt}$. We rearrange to find $e^{(r-\rho)t}/p(0) = c(t)$. Thus $c(0) = 1/p(0)$. Let $c(0) = c_0$ so $c(t) = c_0 e^{(r-\rho)t}$.

Returning to the state equation, it now follows that $\dot{k} = -c_0 e^{-\rho t}$. Integrating, we obtain $k(t) - k(0) = c_0 e^{-\rho t}/\rho - c_0/\rho$. Since $k(0) = W$,

$$k(t) = W + \frac{c_0 e^{-\rho t}}{\rho} - \frac{c_0}{\rho}.$$

The transversality condition says that $\lim_{t\to\infty} p(t) k(t) = 0$. Now $p(t)$ is constant, so this implies $\lim_{t\to\infty} k(t) = 0$. But that requires $W = c_0/\rho$ or $c(0) = c_0 = \rho W$. At this point we have determined the optimal path: $c(t) = (\rho W) e^{(r-\rho)t}$. As in the discrete time consumer's problem of Example 25.2.2 of my manuscript, consumption grows when $r > \rho$ and shrinks with $r < \rho$. When $r = \rho$, consumption is constant over time at $\rho W$. The corresponding costate path is $p(t) = 1/c_0 = 1/\rho W$.

## 34.14 Solving the One-Sector Growth Model

Now consider the one-sector growth model with $u(t,k,c) = e^{-\rho t}u(c)$ and $g(t,k,c) = f(k) - c$. Here $\mathcal{H}(t,k,c) = e^{-\rho t}u(c) + p[f(k) - c]$. Hamilton's equations (the state and costate equations) become $-\dot{p} = \mathcal{H}_k = pf'(k)$ and $\dot{k} = \mathcal{H}_p = f(k) - c$.

To maximize the Hamiltonian, we focus on interior solutions where $c(t) > 0$ for all times $t$. The Inada condition $u'(0+) = +\infty$ will guarantee that the solution is interior. The solution must satisfy the maximization condition $0 = \mathcal{H}_c = e^{-\rho t}u'(c) - p$. Substituting this in the costate equation, we obtain $-(d/dt)\big(e^{-\rho t}u'(c)\big) = e^{-\rho t}u'(c)f'(k)$, which is precisely what we would obtain from the Euler-Lagrange equations. Expanding, this becomes

$$\rho e^{-\rho t}u'(c) - e^{-\rho t}u''(c)\frac{dc}{dt} = e^{-\rho t}u'(c)f'(k).$$

To solve, we divide by $e^{-\rho t}$ and substitute $c = f(k) - \dot{k}$ from the costate equation, obtaining a second-order equation in $k$:

$$\rho u'(c) - u''(c)[f'(k)\dot{k} - \ddot{k}] = u'(c)f'(k)$$

or

$$u''(c)\ddot{k} - u''(c)f'(k)\dot{k} + (\rho - f'(k)u'(c) = 0.$$

This can be solved for the optimal $k$. We need to pin down $k(0)$ and $\dot{k}(0)$ to do that. The initial condition gives $k(0)$ and the transversality condition will usually admit only one possible $\dot{k}(0)$. Note that if $k(0) = \bar{k}$ defined by $f'(\bar{k}) = \rho$, we have a steady state with solution $k(t) = \bar{k}$.

## 34.15 Production with Storage Cost

Let's consider another example which we *can* explicitly solve. A firm must produce Q units of a product by time T. Let the total produced by time $t$ be denoted $k(t)$. Thus $k(0) = 0$ and $k(T) = Q$ are the constraints.

The firm incurs a marginal cost of production $c$, a cost $y^2$ that depends on the rate of production $y = \dot{k}$, and a storage cost $\mu k$. Total cost is $cQ + \int_0^T(\mu k + y^2)dt$, and must be minimized.

This gives us our objective $-(\mu k + y^2)$. The control is $y$, and the state $k$ evolves according to $\dot{k} = y$. Since $k(T)$ is fixed, the transversality condition is not relevant. The condition that $k(T) = Q$ replaces the transversality condition.

The Hamiltonian is

$$\mathcal{H}(t, k, y, p) = -(\mu k + y^2) + py.$$

This yields costate equation $\dot{p} = \mu$, so $p(t) = p_0 + \mu t$. At the maximum, $p = 2y$, thus $\dot{k} = y = p_0/2 + \mu t/2$. It follows that

$$k(t) = \mu t^2/4 + p_0 t/2 + \alpha.$$

As $k(0) = 0$, the constant $\alpha$ is zero. Further, $k(T) = Q$, so $p_0 = (4Q - \mu T^2)/2T$. This solution only makes sense if $4Q \geq \mu T^2$. Otherwise, we get the nonsensical result that production should be negative in the early time periods.

If $4Q < \mu T^2$, the firm should put off production until the remaining time is $t_0 = \sqrt{4Q/\mu}$, then follow the production path from the previous solution. That is

$$y(t) = \begin{cases} 0 & \text{if } t < t_0 \\ y(t - t_0) & \text{if } t_0 \leq t \leq T \end{cases}$$

where $y(t) = p_0/2 + \mu t^2/2$, as in the preceding paragraph.

## 34.16 Current Value Prices

We originally derived the Hamiltonian using present value or time zero prices $p(t)$. One clue to this is that in the optimal growth problem, $p(t) = e^{-\rho t}u'\big((t)\big)$. Prices equal discounted marginal utility. An alternative approach is to use undiscounted time $t$ prices. These are call *current value prices*.

In the one-sector growth model, and all models with objective $e^{-\rho t}u(c)$, there is a useful transformation that gives Hamilton's equations a simpler form. The transformation is $q = e^{\rho t}p$ and

$$H = e^{\rho t}\mathcal{H} = q_0 u(t,k,c) + q g(t,k,c).$$

Economically, $p(t)$ is the present value or time zero price, while $q(t)$ is the price at time $t$ (the current value price). The state equation then becomes $\dot{k} = H_q$, while the costate is $\dot{q} = \rho q + e^{\rho t}\dot{p} = \rho q - H_k$. The maximization condition $H_c = 0$ is unchanged, and the transversality condition is then written $e^{-\rho t}q(t)k(t) \to 0$.

In our context, Hamilton's equations are $\dot{q} = \rho q - qf'$ and $\dot{k} = f(k) - c$, with maximization condition $u'(c) = q$. The use of current value prices eliminates the dependence of $c$ on $t$. We can eliminate $c$ from the second equation since $c = (u')^{-1}(q)$. This yields $\dot{k} = f(k) - (u')^{-1}(q)$.

## 34.17 Current Value Maximum Principle

We can restate the Maximum Principle in terms of current value prices.

***Current Value Maximum Principle.*** *Suppose* $u$ *and* $g$ *are concave and continuously differentiable. Then differentiable* $k^*$ *and continuous* $c^*$ *solve*

$$\begin{aligned} &\max_c \int_0^\infty e^{-\rho t} u(k,c)\,dt \\ &\text{s.t. } \dot{k} = g(t,k,c) \\ &\qquad c \in A(t) \\ &\qquad k(0) = k_0 \end{aligned}$$

*if and only if there is a differentiable function* $q(t)$ *and constant* $q_0 \geq 0$ *such that:*

(1) $\dot{q} = \rho q - H_k$ *(costate equation)*
(2) $\dot{k} = H_q$ *(state equation)*
(3) $H(t,k^*,c^*,q) \geq H(t,k^*,c,q)$ *for all* $c \in A(t)$
(4) $\lim_{t\to\infty} e^{-\rho t} q(t)k(t) = 0$ *(transversality condition)*
(5) $k^*(0) = k_0$ *(initial condition).*

*where* $H$ *is the current value Hamiltonian,*

$$H = u(k,c) + q(t)g(t,k,c).$$

## 34.18 The Stable Manifold Theorem

Current value prices are particularly helpful when studying behavior about the steady state. An important tool is the Stable Manifold Theorem, which uses a linearized version of the system to determine local stability of steady states.[6 7]

Given a steady state $\bar{\mathbf{x}}$ of the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, we can transform the system to one with steady state at $\mathbf{0}$ by using $\mathbf{y} = \mathbf{x} - \bar{\mathbf{x}}$ and writing $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y} + \bar{\mathbf{x}})$. Then the $\mathbf{y}$ derivative evaluated at $\mathbf{0}$ is $\mathbf{A} = D\mathbf{F}(\bar{\mathbf{x}})$.

***Stable Manifold Theorem.*** *Let $U$ be an open subset of $\mathbb{R}^m$ containing the origin. Suppose $\mathbf{x}$ obeys $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ where $\mathbf{F}\colon U \to \mathbb{R}^m$ is $\mathcal{C}^1$ with $\mathbf{F}(\bar{\mathbf{x}}) = \mathbf{0}$. Let $\mathbf{A} = D\mathbf{F}(\bar{\mathbf{x}})$.*

*If $\mathbf{A}$ has $k > 0$ eigenvalues $\lambda$, counted according to multiplicity, that satisfy $\operatorname{Re}\lambda < 0$. Then $S = \{\mathbf{x} : \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \limsup \ln \|\mathbf{x}(t)\|/t < 0\}$ is a $\mathcal{C}^1$ manifold of dimension $k$.*

*Moreover, if $\mathbf{A}$ has no eigenvalues with $\operatorname{Re}\lambda = 0$, there is a homeomorphism $\boldsymbol{\phi}$ mapping a neighborhood of $\bar{\mathbf{x}}$ to a neighborhood of $\mathbf{0}$ that satisfies $\boldsymbol{\phi}\big(\mathbf{x}(t)\big) = e^{\mathbf{A}t}\boldsymbol{\phi}(\mathbf{x}_0)$.*

**Proof.** See Theorems 6.1 and 7.1 in Hartman (1982, pp. 242–244).

The theorems in Hartman apply to the case where the steady state is at zero, but as we have already seen, it is easy to transform these systems into that form. ∎

Although the homeomorphism $\boldsymbol{\phi}$ is necessarily continuous, it need not be differentiable.

[6] The Stable Manifold Theorem establishes local stability. This applies widely in optimal growth problems. See Scheinkman (1976), who uses the "Stable Manifold Theorem for a Point" from Hirsch and Pugh (1970). Brock and Scheinkman (1976) were also able to show global stability optimal growth problems.

[7] It is well-known that saddlepoint stability, with equal numbers of stable and unstable roots, is typical of optimal growth problems (McKenzie, 1963; Samuelson 1968, 1969a, 1970; Levhari and Liviatan, 1972). Boyd (1989) found that the problem of repeated roots had been overlooked, but used the characteristic polynomial to show that even then, both the root and its reciprocal must have the same multiplicity. The discrete-time version was published in Becker and Boyd (1997, sec. 5.4.4). Liviatan and Samuelson (1969) showed that not all optimal growth problems exhibit saddlepoint stability. Boldrin and Montrucchio (1986) found that even chaotic behavior is possible.

## 34.19 The Stable Manifold in the One-Sector Model

Applying the Stable Manifold Theorem gives us locally a stable manifold S of dimension $m$. Solutions that start on the stable manifold remain on the stable manifold. Further, these solutions converge to the steady state. This tells us that if we have initial data in S (values of the costate variables) that is also in a small enough neighborhood of $\bar{x}$, the solution $x$ converges to $\bar{x}$, otherwise, it leaves the neighborhood.

► **Example 34.19.1: Steady State in the One-sector model.** The current value Hamiltonian is

$$H = u(c) + q[f(k) - c]$$

The costate equation is $\dot{q} = \rho q - H_k = q\big(\rho - f(k)\big)$ while the state equation is $\dot{k} = f(k) - c$. Assuming an interior solution, the condition for maximizing the current value Hamiltonian is $u'(c) = q$. This defines a function $c(q)$ with $u'\big(c(q)\big) = q$. Note that $c'(q) = 1/u''\big(c(q)\big)$.

The steady state has $\dot{k} = 0$, so $c = f(k)$. It also obeys $\dot{q} = 0$, so $\rho = f'(k)$. The latter implies $k = \bar{k}$ where $\bar{k}$ is the unique value obeying $f'(\bar{k}) = \rho$. The corresponding level of consumption is $\bar{c} = f(\bar{k})$, which implies the costate is $\bar{q} = u'(\bar{c})$. If a path $\big(k(t), q(t)\big)$ starts at the steady state $(\bar{k}, \bar{q})$, it will always remain there as $\dot{k} = 0$ and $\dot{q} = 0$.

The steady state is typically plotted in $(k, q)$ space as the intersection of the $\dot{k} = 0$ and $\dot{q} = 0$ loci, as in Figure 34.19.2. We can write the differential system that describes the evolution of optimal paths as

$$\begin{aligned} \dot{k} &= f(k) - (u')^{-1}(q) \\ \dot{q} &= q\big(\rho - f'(k)\big) \end{aligned}$$

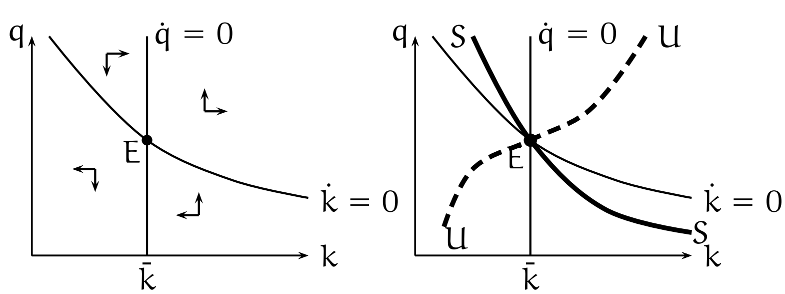


**Figure 34.19.2**: The left diagram shows the steady state (E) at the intersection of the $\dot{k} = 0$ and $\dot{q} = 0$ graphs. The arrows in the four quadrants indicate the direction of motion when the initial conditions lie in that quadrant.

In the right diagram, the heavy line marked SS is the stable manifold. Given $k_0$, it shows the unique initial value of q that results in convergence to the steady state. This is the optimal path. It is the only path obeying both the differential system and the transversality condition. Optimal paths starting on the stable manifold will converge to the steady state. The heavy dashed line is the unstable manifold. Paths starting at any point off the stable manifold asymptotically converge toward the unstable manifold.

◄

## 34.20 The Linearized System and Stability

We can better understand the behavior of this system near the steady state by linearizing it. We approximate the right hand side by its derivative matrix evaluated at the steady state. Let

$$\mathbf{F}(k,q) = \begin{pmatrix} f(k) - (u')^{-1}(q) \\ q\big(\rho - f'(k)\big) \end{pmatrix}$$

The derivative is

$$D_{(k,q)}\mathbf{F} = \mathbf{A}(k,q) = \begin{pmatrix} f'(k) & -1/u''\big(c(q)\big) \\ -qf''(k) & \rho - f'(k) \end{pmatrix}$$

At the steady state $(\bar{k}, \bar{q})$, $c(\bar{q}) = \bar{c}$ and $f'(\bar{k}) = \rho$. Evaluating the derivative at the steady state yields the linearization

$$\begin{pmatrix} \dot{k} \\ \dot{q} \end{pmatrix} = \mathbf{A}(\bar{k},\bar{q}) \begin{pmatrix} k \\ q \end{pmatrix} = \begin{pmatrix} \rho & -1/u''(\bar{c}) \\ -\bar{q}f''(\bar{k}) & 0 \end{pmatrix} \times \begin{pmatrix} k \\ q \end{pmatrix}.$$

We compute the eigenvalues of $\mathbf{A}$ to investigate the dynamics around the steady state. The characteristic equation is

$$\lambda(\lambda - \rho) - \bar{q}f''(\bar{k})/u''(\bar{c}) = 0.$$

Let $\alpha = \bar{q}f''(\bar{k})/u''(\bar{c}) > 0$. Then $\lambda^2 - \rho\lambda - \alpha = 0$ so

$$\lambda = \rho/2 \pm \frac{1}{2}\sqrt{\rho^2 + 4\alpha}.$$

It is evident that the square root is greater than $\rho$, so there is one positive root and one negative root, indicating that the steady state is saddlepoint stable.

Moreover, the positive root is greater than $\rho$, so solutions corresponding to the positive root violate the transversality condition. The solutions can only include a term with the negative root.

We now consult the Stable Manifold Theorem. For each value of $k(0)$ in some neighborhood of $\bar{k}$, there is a unique $q(0)$ near $\bar{q}$ that is on the stable manifold. The optimal trajectory of the original system then follows the stable manifold in toward the steady state. Other values of $q(0)$ define trajectories that violate the transversality condition. They are not optimal and do not converge to the steady state.

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