

Constrained Optimization Survival Guide

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August 2003

Maximize utility, minimize costs, maximize profits, maximize welfare, minimize the loss function—these are all typical economics problems. Indeed, Lionel Robbins defined economics as the study of the allocation of scarce resources among competing ends. All of the above problems typically involve some notion of scarcity, some constraint. There are limits on what is technically feasible (production possibilities) or on what is affordable. This survival guide aims at showing you the basic techniques for solving such problems. These basics will help you in your other classes. The fancier stuff can wait until later in this course.

The main technique for solving constrained optimization problems is the method of Lagrange multipliers. The problem is to maximize (or minimize) a function $u(c)$ (the *objective*), under the constraint that $g_i(c) = 0$ for $i = 1, \dots, m$. We first form the Lagrangian $\mathcal{L} = u(c) + \sum_{i=1}^m \lambda_i g_i(c)$. The new variables λ_i are called *Lagrange multipliers*. When u and the g_i are differentiable, the optimum must obey the *first-order conditions*: $\partial \mathcal{L} / \partial c_i = 0$. Not everything that solves the first-order conditions is necessarily an optimum, but it usually is in well-behaved economic problems. In fact, in most economic problems the solution to the first-order equations is not only an optimum, but is the right kind of optimum (maximum when we want to maximize, minimum when we want to minimize). In the second half of this course we will develop criteria for distinguishing maxima from minima and from non-optimal solutions.

Utility Maximization

The standard consumer's problem is to maximize utility under a budget constraint. This is exactly the type of problem that Lagrange multipliers are designed to deal with.

Example 1: Utility is $u(x, y) = xy$. The price of x is \$5. The price of y is \$3. We have \$15 to spend. Thus we maximize utility under the budget constraint $5x + 3y = 15$. Set up the

Lagrangian $\mathcal{L} = xy + \lambda(5x + 3y - 15)$. There are two first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = y + 5\lambda = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial y} = x + 3\lambda = 0.$$

We next eliminate λ from the equations by rewriting $x/3 = -\lambda = y/5$. Thus $x = 3y/5$. Substituting this into the budget constraint, we find $15 = 5x + 3y = 3y + 3y = 6y$. Thus $y = 5/2$. It follows that $x = 3y/5 = 3/2$. Our solution is $(x, y) = (3/2, 5/2)$.

Example 2.: Utility is $u(x, y) = \ln x + \ln y$. We maximize utility subject to the budget constraint $3x + 6y = 36$. The Lagrangian is $\mathcal{L} = \ln x + \ln y + \lambda(36 - 3x - 6y)$ (notice that we may write the constraint in two ways). The first-order conditions are $\partial \mathcal{L} / \partial x = 1/x - 3\lambda = 0$ and $\partial \mathcal{L} / \partial y = 1/y - 6\lambda = 0$. Thus $2/x = 6\lambda = 1/y$. It follows that $2y = x$. Substitute in the budget constraint to find $36 = 3x + 6y = 6y + 6y = 12y$. Thus $y = 3$ and so $x = 6$.

Example 3.: Utility is $u(x, y) = -1/x - 1/y$. The budget constraint is $p_x x + p_y y = I$. We follow the same steps. (1) Form the Lagrangian $\mathcal{L} = -1/x - 1/y + \lambda(p_x x + p_y y - I)$. (2) Obtain the first-order conditions $1/x^2 + \lambda p_x = 0$ and $1/y^2 + \lambda p_y = 0$. (3) Eliminate λ by setting $1/(p_x x^2) = -\lambda = 1/(p_y y^2)$, so $x^2 = (p_y/p_x)y^2$. (4) Solve for x (or y), $x = \pm(p_y/p_x)^{1/2}y$. In this case consumption of goods must be non-negative, so only the positive sign makes sense. Thus $x = y(p_y/p_x)^{1/2}$. (5) Substitute your expression for x in the budget constraint and solve for y and then x . Here, $I = p_x x + p_y y = (p_x p_y)^{1/2}y + p_y y = p_y^{1/2}(p_x^{1/2} + p_y^{1/2})y$, so $y = I/p_y^{1/2}(p_y^{1/2} + p_x^{1/2})$. It follows that $x = y(p_y/p_x)^{1/2} = I/p_x^{1/2}(p_y^{1/2} + p_x^{1/2})$. These are the *consumer demand functions*

$$x(p_x, p_y, I) = \frac{I}{p_x^{1/2}(p_y^{1/2} + p_x^{1/2})} \quad \text{and} \quad y(p_x, p_y, I) = \frac{I}{p_y^{1/2}(p_y^{1/2} + p_x^{1/2})}.$$

Notice that both demand functions obey the Law of Demand. When the price of x rises, quantity of x demanded falls (and similarly for y). Moreover, both goods are normal. When income rises, demand also rises. Finally, the goods are complements because an increase in the price of one leads to a fall in demand for the other.

Usually, you can solve utility maximization problems by following the steps above. This is not the only way to do it, but it is a method that works.

Cost Minimization

When studying the behavior of a firm, it is useful to examine the problem finding the minimum cost to produce a given quantity q . This is a problem that interests both competitive

firms and monopolies. The minimum cost is the *cost function* $C(q)$. Typically, we are given a production function that tells us how much can be produced from a each combination of factor inputs. We are also told the price of the factors of production.

Example 4.: The price of x is \$3 while the price of y is \$2. The production function is defined by $q = xy$. We must find the cost-minimizing factor inputs and the cost of production.

Set up the Lagrangian $\mathcal{L} = 3x + 2y + \lambda(q - xy)$. The first-order conditions are $3 - \lambda y = 0$ and $2 - \lambda x = 0$. Thus $3/y = \lambda = 2/x$, and so $x = 2y/3$. Now use the constraint $q = xy = 2y^2/3$. We find $y = \sqrt{3q/2}$ (the negative root doesn't make economic sense). This implies $x = \sqrt{2q/3}$. These are the optimal factor inputs (factor demands). The cost of x is $3x = \sqrt{6q}$ and the cost of y is $2y = \sqrt{6q}$. Adding the costs of the factors together yields the cost of producing q , $C(q) = 2\sqrt{6q}$.

Example 5.: The price of x is w_x while the price of y is w_y . The production function is given by $q = \sqrt{x} + \sqrt{y}$. We will derive the cost as a function of quantity produced q and factor prices (w_x, w_y) .

The Lagrangian is $\mathcal{L} = w_x x + w_y y + \lambda(q - \sqrt{x} - \sqrt{y})$. The first-order conditions are $w_x - \lambda/(2\sqrt{x}) = 0$ and $w_y - \lambda/(2\sqrt{y}) = 0$. Thus $2w_x\sqrt{x} = \lambda = 2w_y\sqrt{y}$, so $\sqrt{x} = (w_y/w_x)\sqrt{y}$. Now we substitute this into the constraint $q = \sqrt{x} + \sqrt{y}$ to find $q = (w_y/w_x)\sqrt{y} + \sqrt{y}$. Solving for y , we obtain $y = q^2 w_x^2 / (w_x + w_y)^2$. This means $x = q^2 w_y^2 / (w_x + w_y)^2$. These are the *conditional factor demand functions*. These conditional factor demands also obey the Law of Demand. In this case, the factor demands are also affected by changes in the price of the other good. These factors are substitutes—an increase in the price of one increases demand for the other. Finally, we multiply the factor demands x and y by their prices and add to obtain the cost function

$$C(q, w_x, w_y) = w_x x + w_y y = \frac{q^2 w_y^2}{(w_x + w_y)}.$$