Micro II Midterm, October 15, 2007

1. There are two consumer and two goods. The indirect utility functions of the consumers are $v_1(\mathbf{p}, m) = (p_1+m)/(p_1+p_2)$ and $v_2(\mathbf{p}, m) = (p_2+m)/(p_1+p_2)$. If these are the only consumers, is market demand a function of prices and aggregate income? Or does the income distribution affect market demand? Explain.

   Answer: Version 1: Both consumer's have Gorman form indirect utility. Moreover, the term multiplying income is the same for both. These implies that market demand is a function of prices and aggregate income.

   Version 2: We use Roy's Identity to compute demand. Consumer 1's demand is $\mathbf{x}^1(\mathbf{p}, m_1) = (p_1+p_2)^{-1}(m_1 - p_2, m_1 + p_1)$ and consumer 2's demand is $\mathbf{x}^2(\mathbf{p}, m_2) = (p_1+p_2)^{-1}(m_2 - p_1, m_2 + p_2)$, yielding aggregate demand $(p_1+p_2)^{-1}(m - p_1 - p_2, m + p_1 + p_2)$, which clearly depends only on $\mathbf{p}$ and aggregate income $m = m_1 + m_2$.

2. A firm using two inputs has production function $f(\mathbf{z}) = (z_1 z_2)^{1/3}$. Factor prices are $\mathbf{w} = (w_1, w_2) \gg \mathbf{0}$.

   a) Suppose the inputs are goods 1 and 2 and the output is good 3. Find the production set $Y \subset \mathbb{R}^3$.

   Answer: The input is $z_1 = -y_1$ and $z_2 = -y_2$. Since output is y_3, the production function tells us that $y_3 \le f(-y_1, -y_2) = (y_1 y_2)^{1/3}$. We know inputs must be negative ($y_1 \le 0$, $y_2 \le 0$), so the production set is $Y = \{\mathbf{y} \in \mathbb{R}^3 : y_1 \le 0, y_2 \le 0, y_3 \le (y_1 y_2)^{1/3}\}$.

   b) Given q, find the conditional factor demands.

   Answer: Since the production function is Cobb-Douglas with equal weights, we know that $w_1 z_1 = w_2 z_2$, so $z_2 = (w_1/w_2) z_1$. Now substitute in $q = f(\mathbf{z})$ to obtain $(w_1/w_2)^{1/3} z_1^{2/3} = q$, so $z_1 = (w_2/w_1)^{1/2} q^{3/2}$ and $z_2 = (w_1/w_2)^{1/2} q^{3/2}$, which are the conditional factor demands $z_\ell(\mathbf{w}, q)$.

   c) Compute the cost function.

   Answer: Cost is $c(\mathbf{w}, q) = \mathbf{w}\cdot\mathbf{z}(\mathbf{w}, q) = 2\sqrt{w_1 w_2}\, q^{3/2}$.

3. Suppose utility has the additive separable form $U(\mathbf{x}) = u_1(x_1) + u_2(x_2) + u_3(x_3)$ where the u_ℓ are twice continuously differentiable on $\mathbb{R}_+$ with $u_\ell' > 0$ and $u_\ell'' < 0$. Moreover, $\lim_{x\to 0} u_\ell'(x) = +\infty$ for $\ell = 1, 2, 3$. Consider the consumer's utility maximization problem given income $m > 0$ and prices $\mathbf{p} \gg \mathbf{0}$.

   a) Find the first-order conditions and show that there are no corner solutions.

   Answer: The Lagrangian is $\mathcal{L} = u_1 + u_2 + u_3 + \lambda(m - \mathbf{p}\cdot\mathbf{x}) + \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3$. The first-order conditions are then $u_\ell'(x_\ell) + \mu_\ell = \lambda p_\ell$ for $\ell = 1, 2, 3$. The Inada condition $\lim_{x\to 0} u_\ell'(x) = +\infty$ insures that $x_\ell = 0$ cannot solve the first-order equations. Thus $x_\ell > 0$ and $\mu_\ell = 0$ by complementary slackness. The first-order conditions now read $u_\ell'(x_\ell) = \lambda p_\ell$.

   b) Eliminate the remaining multiplier from the first-order conditions. Then show that $\partial x_\ell/\partial m$ has the same sign for all ℓ.

   Answer: As usual, we eliminate the multiplier for the budget constraint by dividing the first order conditions. Thus $u_k'(x_k)/u_\ell'(x_\ell) = p_k/p_\ell$. We re-write this as $p_\ell u_k'(x_k) = p_k u_\ell'(x_\ell)$. Then take the derivative with respect to m, obtaining $p_\ell u_k''(x_k)\partial x_k/\partial m = p_k u_\ell''(x_\ell)\partial x_\ell/\partial m$. Since the second derivatives are negative, $\partial x_\ell/\partial m$ and $\partial x_k/\partial m$ have the same sign.

   c) Now show that all goods are normal.

   Answer: We use Walras' Law, $m = \mathbf{p}\cdot\mathbf{x}(\mathbf{p}, m)$. Take the derivative with respect to m to find

$1 = \sum p_k \partial x_k/\partial m$. Since all of the $\partial x_k/\partial m$ have the same sign, that sign must be positive to satisfy the equation. In other words, all goods are normal.

4. Suppose the indirect utility is

$$v(\mathbf{p}, m) = \begin{cases} \ln \frac{m}{p_2} & \text{if } p_1 \geq m \\ \frac{m-p_1}{p_1} + \ln \frac{p_1}{p_2} & \text{if } p_1 \leq m \end{cases}$$

a) Compute the expenditure function.

Answer: When $p_1 \geq m$, we use duality to write $\bar{u} = \ln(e(\mathbf{p}, \bar{u})/p_2)$, so $e(\mathbf{p}, \bar{u}) = p_2 e^{\bar{u}}$. The condition $p_1 \geq m$ becomes $p_1 \geq e(\mathbf{p}, \bar{u}) = p_2 e^{\bar{u}}$, or $\ln(p_1/p_2) \geq \bar{u}$.

We then use duality on the case $p_1 \leq m$ to obtain $\bar{u} = e(\mathbf{p}, \bar{u})/p_1 - 1 + \ln(p_1/p_2)$. This yields $e(\mathbf{p}, \bar{u}) = p_1(1 + \bar{u}) - p_1 \ln(p_1/p_2)$. The condition $p_1 \leq m$ translates to $p_1 \leq e(\mathbf{p}, \bar{u}) = p_1[1 + \bar{u} - \ln(p_1/p_2)$. Thus $1 \leq 1 + \bar{u} - \ln(p_1/p_2)$, equivalently, $\ln(p_1/p_2) \leq \bar{u}$.

Summing up,

$$e(\mathbf{p}, \bar{u}) = \begin{cases} p_2 e^{\bar{u}} & \text{if } \ln(p_1/p_2) \geq \bar{u} \\ p_1(1 + \bar{u}) - p_1 \ln(p_1/p_2) & \text{if } \ln(p_1/p_2) \leq \bar{u}. \end{cases}$$

FYI. The utility function here is $x_1 + \ln x_2$.

b) Compute the Hicksian demand function.

Answer: We use the Shephard-McKenzie Lemma to find

$$\mathbf{h}(\mathbf{p}, \bar{u}) = \begin{cases} (0, e^{\bar{u}}) & \text{if } \ln(p_1/p_2) \geq \bar{u} \\ (\bar{u} - \ln \frac{p_1}{p_2}, \frac{p_1}{p_2}) & \text{if } \ln(p_1/p_2) \geq \bar{u} \end{cases}$$