Micro I Midterm, February 26, 2008

1. Suppose the expenditure function is $e(\mathbf{p}, \bar{u}) = p_1 + \sqrt{p_1 p_2} + \bar{u}(p_1 + p_2)$ for $\bar{u} \geq 0$.

   a) Find the Hicksian demand functions.

   Answer: Using the Shephard-McKenzie Lemma, we find $h_1(\mathbf{p}, \bar{u}) = 1 + \bar{u} + (1/2)\sqrt{p_2/p_1}$ and $h_2(\mathbf{p}, \bar{u}) = \bar{u} + (1/2)\sqrt{p_1/p_2}$.

   b) Find the indirect utility function.

   Answer: We use the duality relation $m = e(\mathbf{p}, v(\mathbf{p}, inc))$ to write $m = p_1 + \sqrt{p_1 p_2} + v(\mathbf{p}, m)(p_1 + p_2)$. Then solve for v to obtain $v(\mathbf{p}, m) = (m - p_1 - \sqrt{p_1 p_2})/(p_1 + p_2)$.

2. Suppose preferences on $\mathfrak{X} = \mathbb{R}^2_+$ are described by the utility function

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 = 0. \end{cases}$$

   a) Are these preferences convex?

   Answer: Yes. Consider $u(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}')$.

   If $\alpha x_1 + (1-\alpha)x_1' = 0$, then $x_1 = x_1' = 0$, and $u(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') = 0 = \alpha u(\mathbf{x}) + (1-\alpha)u(\mathbf{x}'))$.

   If $\alpha x_1 + (1-\alpha)x_1' > 0$, then $u(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') = \alpha(x_1 + x_2) + (1-\alpha)(x_1' + x_2') \geq \alpha u(\mathbf{x}) + (1-\alpha)u(\mathbf{x}')$.

   Either way, $u(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') \geq \alpha u(\mathbf{x}) + (1-\alpha)u(\mathbf{x}')$, so u is concave and the preference order is convex.

   b) Are these preferences separable relative to the partition $\mathcal{P} = \{\{1\}, \{2\}\}$?

   Answer: No. Let $x_2 > x_2'$. For $x_1 > 0$, $(x_1, x_2) \succ (x_1, x_2')$ but $(0, x_2) \sim (0, x_2')$.

   c) For what prices $\mathbf{p} \gg \mathbf{0}$ and incomes $m > 0$ does the consumer's problem have a solution?

   Answer: If we consider the related linear preferences $v(\mathbf{x}) = x_1 + x_2$, we know the optimum is $(0, m/p_2)$ when $p_1 > p_2$; $(m/p_1, 0)$ when $p_1 < p_2$, and any $\mathbf{x} \in \mathbb{R}^2_+$ with $p_1 x_1 + p_2 x_2 = m$ when $p_1 = p_2$. The first case gives utility zero with the actual preferences. There is no solution in this case $(p_1 > p_2)$. The last case also needs modification to rule out $x_1 = 0$. Thus

$$\mathbf{x}(\mathbf{p}, m) = \begin{cases} \text{no solution} & \text{when } p_1 > p_2 \\ \{\mathbf{x} \in \mathbb{R}^2_+ : x_1 > 0, p_1 x_1 + p_2 x_2 = m\} & \text{when } p_1 = p_2 \\ (m/p_1, 0) & \text{when } p_1 < p_2. \end{cases}$$

3. There are two consumers. Consumer 1 has utility $u_1(x_1, x_2) = x_1^{3/4} x_2^{1/4}$ and wealth $m_1 > 0$. Consumer 2 has utility $u_2(x_1, x_2) = x_1^{1/4} x_2^{3/4}$ and wealth $m_2 > 0$.

   a) Find the individual demand functions $\mathbf{x}^i(\mathbf{p}, m_i)$.

   Answer: These Cobb-Douglas utility functions have demands $\mathbf{x}^1(\mathbf{p}, m_1) = (m_1/4)(3/p_1, 1/p_2)$ and $\mathbf{x}^2(\mathbf{p}, m_2) = (m_2/4)(1/p_1, 3/p_2)$.

   b) Find aggregate demand $\sum_{i=1}^2 \mathbf{x}^i(\mathbf{p}, m_i)$.

   Answer: Aggregate demand is $\left(\frac{3m_1 + m_2}{4p_1}, \frac{m_1 + 3m_2}{4p_2}\right)$.

c) Either show that aggregate demand can be written as a function of $\mathbf{p}$ and $m = m_1 + m_2$ or provide an example to show it cannot be done.

Answer: It cannot be done. Consider $(m_1, m_2) = (0, 1)$ and $(m_1, m_2) = (1, 0)$. Although aggregate income is the same, they yield different aggregate demands.

d) Suppose $m_1 = m/3$ and $m_2 = 2m/3$. Does aggregate demand obey the Law of Demand?

Answer: Because preferences are homothethic, the Law of Demand holds.

Alternatively, one can compute aggregate demand $\mathbf{x}(\mathbf{p}, m) = (5m/12p_1, 7m/12p_2)$. The requirement that $(\mathbf{p}'-\mathbf{p})\cdot(\mathbf{x}(\mathbf{p}', m)-\mathbf{x}(\mathbf{p}, m)) \leq 0$ is equivalent to $D_{\mathbf{p}}\mathbf{x}$ being negative semidefinite. Since

$$D_{\mathbf{p}}\mathbf{x} = \frac{m}{12}\begin{bmatrix} -\frac{5}{p_1^2} & 0 \\ 0 & -\frac{7}{p_2^2} \end{bmatrix},$$

that condition is satisfied.

A third method is the brute force calculation that

$$(\mathbf{p}' - \mathbf{p})\cdot\big(\mathbf{x}(\mathbf{p}', m) - \mathbf{x}(\mathbf{p}, m)\big) = -\frac{5m(p_1' - p_1)^2}{12p_1p_1'} - \frac{7m(p_2' - p_2)}{12p_2p_2'} \leq 0,$$

with equality only if $\mathbf{p}' = \mathbf{p}$.

4. A firm has production set $Y = \{\mathbf{y} \in \mathbb{R}^3 : y_1, y_2 \leq 0, y_1 + y_3 - \sqrt{-y_2} \leq 0\}$.

a) Does the technology obey constant returns to scale?

Answer: No. $(-1, -1, 2) \in Y$ but $(-2, -2, 4) \ni Y$.

b) Find the supply (net output) function $\mathbf{y}(\mathbf{p})$.

Answer: We form the Lagrangian $p_1y_1 + p_2y_2 + p_3(-y_1 + \sqrt{-y_2}) - \mu_1y_1 - \mu_2y_2$. This yields first-order conditions $p_1 = p_3 + \mu_1$ and $p_2 = p_3/2\sqrt{-y_2} + \mu_2$. Note that there is no solution if $p_3 > p_1$. The first-order conditions do not permit $y_2 = 0$, so we may assume $y_2 < 0$. Complementary slackness implies $\mu_2 = 0$. Thus $y_2 = -p_3^2/4p_2^2$. If $p_1 > p_3$, $\mu_1 > 0$, so $y_1 = 0$ by complementary slackness. In that case $y_3 = p_3/2p_2$. If $p_1 = p_3$, any $y_1 \leq 0$ is ok and $y_3 = -y_1 + \sqrt{-y_2} = -y_1 + p_3/2p_2$. Summing up,

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \text{no solution} & \text{when } p_1 < p_3 \\ \left\{\left(y_1, -\frac{p_3^2}{4p_2^2}, -y_1 + \frac{p_3}{2p_2}\right)\right\} & \text{when } p_1 = p_3 \\ \left(0, -\frac{p_3^2}{4p_2^2}, \frac{p_3}{2p_2}\right) & \text{when } p_1 > p_3 \end{cases}$$

c) Find the profit function $\pi(\mathbf{p})$.

Answer: Using part (b), we find the profit is

$$\pi(\mathbf{p}) = \begin{cases} p_3^2/4p_2 & \text{when } p_1 \geq p_3 \\ +\infty & \text{when } p_1 < p_3. \end{cases}$$