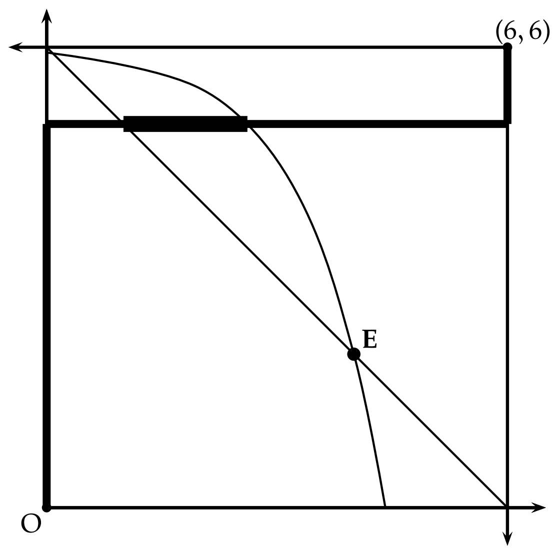
Micro I Final, April 22, 2008

1. Consider a two-person, two-good exchange economy. Consumer 1 has endowment $(4,2)$ and utility $u_1(x_1,x_2) = x_1 + x_2$. Consumer 2 has endowment $(2,4)$ and utility $u_2(x_1,x_2) = x_1 + \ln x_2$.

   a) Find all Pareto optima.

   Answer: We can find all interior Pareto optima by equating the marginal rates of substitution. This yields $1 = x_2^2$. This means $x_2^1 = 5$ and $x_2^2 = 1$ for the interior Pareto optima. The only other conditions are the constraints $x_1^1, x_1^2 \geq 0$ and $x_1^1 + x_1^2 = 6$.

   There are two types of Pareto optima on the boundary of the Edgeworth box. Those in the upper right where consumer one has higher utility than $u_1(6,5)$ (and consumer two has less that $u_2(0,1)$) and those on the lower left where consumer 2 has higher utility that $u_2(6,1)$. These points comprise the rest of the Pareto optima. This is illustrated by the heavy line in the diagram.



   b) Find the core.

   Answer: In a two-person exchange economy, an allocation is in the core if and only if it is individually rational and Pareto optimal. Individual rationality requires $x_1^1 + x_2^1 = u_1(\mathbf{x}^1) =\geq u_1(\boldsymbol{\omega}^1) = 6$ and $x_1^2 + \ln x_2^2 = u_2(\mathbf{x}^2) \geq u_2(\boldsymbol{\omega}^2) = 2 + \ln 4 \approx 3.386$.

   The line and curve through the endowment point E demarcate the individually rational points. The core is the portion of the Pareto set that lies above the diagonal line and below the curve in the diagram. It is the set of points $\{\big((x,5),(6-x,1)\big) : 1 \leq x \leq 4\ln 4\}$, which gets an extra heavy black line. Note that $4 - \ln 4 \approx 2.614$.

   c) Find all Walrasian equilibria.

   Answer: Any Walrasian equilibrium must be in the core. Since we already know the core guarantees positive consumption of both goods by both consumers, we can eliminate corner solutions from consideration.

Due to linear utility, consumer one will only have interior solutions when $p_1 = p_2$, so we may take $p_1 = p_2 = 1$. The (normalized) equilibrium price vector is $\mathbf{p} = (1,1)$.

Now set $1 = \text{MRS}^2 = x_2^2$. Consumer two's budget constraint is $x_1^2 + x_2^2 = 6$, so $x_1^2 = 5$. By market clearing, $x_1^1 = 1$ and $x_1^2 = 5$, which is on consumer one's budget line and indifference curve.

It follows that the equilibrium is $\mathbf{p} = (1,1)$, $\mathbf{x}^1 = (1,5)$ and $\mathbf{x}^2 = (5,1)$.

2. Suppose the expenditure function is $e(\mathbf{p},\bar{u}) = 2\bar{u}\sqrt{p_1p_2}$.

   a) Find the Hicksian demands.

   Answer: The Shephard-McKenzie lemma yields the demands,

$$\mathbf{h}(\mathbf{p},\bar{u}) = D_{\mathbf{p}}e(\mathbf{p},\bar{u}) = \bar{u}\left(\sqrt{\frac{p_2}{p_1}}, \sqrt{\frac{p_1}{p_2}}\right).$$

   b) Find the indirect utility function.

   Answer: We use the duality relation $m = e\big(\mathbf{p}, v(\mathbf{p},m)\big) = 2v(\mathbf{p},m)\sqrt{p_1p_2}$. It follows that $v(\mathbf{p},m) = m/(2\sqrt{p_1p_2})$.

   c) Find the original utility function.

   Answer: As the relative price changes, the Hicksian demands $\mathbf{h}(\mathbf{p},\bar{u})$ traces out the indifference curve for utility level $\bar{u}$. We must eliminate the relative price from the Hicksian demands. Now $p_2/p_1 = [h_1(\mathbf{p},\bar{u})/\bar{u}]^2 = [h_2(\mathbf{p},\bar{u})/\bar{u}]^{-2}$. It follows that $\bar{u}^2 = h_1h_2$, so the utility function is $u(\mathbf{x}) = \sqrt{x_1x_2}$.

3. Consider a two-agent, two-good economy production economy where utility is $u_1(x_1,x_2) = (x_1)^{1/2}(x_2)^{1/2}$ and $u_2(x_1,x_2) = \sqrt{x_1} + \sqrt{x_2}$. Endowments are $\boldsymbol{\omega}^1 = (3,0)$ and $\boldsymbol{\omega}^2 = (4,1)$. There is one firm with production set $Y = \{(y_1,y_2) : y_1 \le 0, y_2 \le -y_1/2\}$. Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

   Answer: Since utility is Cobb-Douglas, the equilibrium prices of both goods must be positive, and the demand for both goods will also be positive. We set $p_1 = 1$ and $p_2 = p$.

   Consumer 1 has equal-weighted Cobb-Douglas utility and income $m_1 = 3$, so 1's demand is $\mathbf{x}^2(p) = (3/2, 3/2p)$. Consumer 2's demand must be calculated. Corner solutions are ruled out by infinite marginal utility. The first-order conditions yield $(x_2/x_1)^{1/2} = 1/p$, so $x_2 = x_1/p^2$. Substituting in the budget constraint yields $m_2 = x_1 + px_2 = x_1 + x_1/p = (1+p)x_1/p$. Thus $x_1 = pm_2/(1+p)$. Now $m_2 = 4+p$, so $\mathbf{x}^2(p) = (4+p)\big(p/(1+p), 1/p(1+p)\big)$.

   Suppose good 2 is produced by the firm. The firm's zero profit condition then implies $2p_1 = p_2$. By our normalization, $\mathbf{p} = (1,2)$. Then demands are $\mathbf{x}^1 = (3/2, 3/4)$ and $\mathbf{x}^2 = (4,1)$. Market clearing requires $\mathbf{x}^1 + \mathbf{x}^2 = \boldsymbol{\omega} + \mathbf{y}$. Thus $\mathbf{y} = (-3/2, 3/4)$, which is feasible.

   We now consider whether there is an equilibrium where the firm does not produce. This requires $p < 2$. Demand for good 1 is $3/2 + (4+p)p/(1+p) = 7$ by market clearing (here $\mathbf{y} = \mathbf{0}$). This implies $2p^2 - 3p - 11 = 0$. The solution is $p = 3/4 + \sqrt{97}/4 \approx 3.2 > 2$. But this contradicts the shut-down condition $p < 2$. There is only the one equilibrium.

4. Suppose a consumer has discount factor $0 < \delta < 1$ and period utility function $u(c) = \ln c$. The consumer has wealth $W > 0$ and faces prices $p_t = p > 0$ for all times t. Find the optimal consumption path.

   Answer: Since the marginal utility of zero consumption is infinite, consumption will always be positive (unless wealth is zero). The first-order conditions are $\delta u'(c_{t+1})/u'(c_t) = p_{t+1}/p_t$. This becomes

$\delta c_t/c_{t+1} = p/p = 1$, so $c_{t+1} = \delta c_t$. It follows that $c_t = \delta^t c_0$. The budget constraint is $W = \sum_t pc_t = \sum_t p\delta^t c_0 = pc_0/(1-\delta)$. Thus $c_0 = (1-\delta)W/p$ and $c_t = (1-\delta)\delta^t W/p$.

If you don't recall how to sum the infinite series, let $S = \sum_{t=0}^{\infty} \delta^t$. Then $1 + \delta S = \delta^0 + \sum_{t=1}^{\infty} \delta^t = S$. It follows that $S = (1-\delta)^{-1}$. This requires $|\delta| < 1$ for the summation to make converge.