Micro I Final, April 26, 2012

1. Suppose that a gambler discounts future utility at rate $\rho > 0$ per period, yielding discount factor $\delta = 1/(1+\rho) < 1$ and has felicity function $u(c) = c^{1-\sigma}/(1-\sigma)$ where $\sigma > 0$ and $\sigma \neq 1$. In that case, the sum of disconted felicity can be regarded as a von Neumann-Morgenstern expected utility function.

   Consider a gamble that pays off as follows. The payoff starts at \$1 and a fair coin is flipped. If the coin turns up "heads", the game ends and the gambler gets the current payoff. If the coin turns up "tails", the gambler gets nothing this period and the payoff is doubled. Keep repeating until the game ends.

   a) How much will the gambler be willing to pay to play this game?

   b) How does the gambler's willingness to pay change as σ and ρ vary?

   Answer:

   a) The probability of a payoff in period 0 is $\frac{1}{2}$, in period 2 is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, etc. Thus the probability of a payoff at time t, $t = 0, 1, 2, \ldots$, is $\pi_t = 1/2^{t+1}$. If the payoff occurs at time t, the amount of the payoff is $c_t = 2^t$, yielding felicity $u(u_t) = 2^{t(1-\sigma)}/(1-\sigma)$. If the payoff doesn't occur, felicity is zero. The felicity must be discounted to the present, yielding $\delta^{t-1}2^{t(1-\sigma)}/(1-\sigma)$. This is the discounted utility obtained if the payoff occurs at time t. To find expected utility, we multiply by the probability of the payoff at time t and sum. In other words, we sum $\delta^{t-1}2^{-t\sigma}/2(1-\sigma)$. This gives

   $$EU = \frac{1}{2(1-\sigma)} \sum_{t=0}^{\infty} \delta^{t-1} 2^{-t\sigma} = \frac{1}{2(1-\sigma)(2^\sigma - \delta)}.$$

   We can use the certainty equivalent e to measure willingness to pay for the gamble. We find e by setting $u(e) = EU$. Then $e(\sigma, \delta) = (2^{\sigma+1} - 2\delta)^{1/(\sigma-1)}$. This is the maximum that the gambler would be willing to pay.

   b) As ρ increases, δ decreases. Since $2(2^\sigma - \delta)$ is raised to a positive power when $\sigma > 1$ and negative power when $\sigma < 1$, the willingness to pay, e, will increase when $\sigma > 1$ and decrease when $\sigma < 1$.

   The case of σ varying is more complex than anticipated. This can be answered by considering $\ln e = (\sigma - 1)^{-1} \ln(2^{\sigma+1} - 2\delta)$. and taking its derivate with respect to σ.

2. Consider a two-person, two-good exchange economy. Consumer 1 has utility $u_1(\mathbf{x}^i) = \min\{x_1^1, x_2^1\}$ and consumer 2 has utility $u_2(\mathbf{x}^2) = x_1 + 2x_2$. The endowments are $\boldsymbol{\omega}^1 = (6, 1)$ and $\boldsymbol{\omega}^2 = (0, 2)$.

   a) Find all the equilibria of this economy.

   b) Find all Pareto optima.

   Answer:

   a) Here we exploit the facts that the equilibrium must be Pareto optimal and lie on the budget line. If consumer two is not at a corner point, the price vector must be $\mathbf{p} = (1, 2)$ (or a multiple). Then incomes are $m^1 = 8$ and $m^2 = 4$. Consumer one demands $\mathbf{x}^1 = (8/3, 8/3)$ and $\boldsymbol{\omega} - \mathbf{x}^1 = (10/3, 1/3)$ is worth \$4, so $\mathbf{x}^2$ is a demand point for consumer two at these prices. It follows that we have an equilibrium at $\mathbf{p} = (1, 2)$.

   Note that if either price is zero, consumer two may obtain infinite utility, so both prices must be positive. Now if $2p_1 > p_2$, $x_1^2 = 0$. Then $x_1^1 = (6p_1 + p_2)/(p_1 + p_2) < 6$. This excess supply means $p_1 = 0$, which is impossible.

   Similarly, if $2p_1 < p_2$, $x_2^2 = 0$. Then $x_2^1 = (6p_1 + p_2)/(p_1 + p_2)$, which must be 3 since we cannot have excess supply. This implies $p_1 = (2/3)p_2$, which is impossible since $2p_1 < p_2$.

   b) If $x_1^1 \neq x_2^1$, consumer one has an excess of one of the goods. Giving the excess to consumer two is a Pareto improvement. This leaves $P = \{(\mathbf{x}^1, \mathbf{x}^2) : x^1_1 = x^1_2, x^2_1 = 6 - x^1_1, x^2_2 = 3 - x^1_1, 0 \le x^1_1 \le 3\}$ as the only possible Pareto optima. Since there is a utility tradeoff between the points of P, all points in P are Pareto optima.

3. Consider the Ramsey (one-sector) model of capital accumulation with production function $f(a) = 2a$ and felicity function $u(c) = \ln c$. The discount factor δ obeys $0 < \delta < 1$. Let $x > 0$ be the initial stock (there is no other endowment).

   a) Find the optimal paths of consumption and capital accumulation.

   b) Find the corresponding (Malinvaud) price path p_t.

   c) Does the transversality condition hold: $\lim_{t\to\infty} p_t a_t = 0$.

   Answer:

   a) Let p_t be the sequence of prices, so the consumer's budget constraint is $p_0 x = \sum_{t=0}^{\infty} p_t c_t$. The firm's maximization problem shows that $p_t = 2^{-t} p_0$ (the firm maximizes $p_{t+1} f(a_t) - p_t a_t$). The consumer's first-order conditions are $\delta^{t+1} u'(c_{t+1})/ = \delta^t u'(c_t)/p_t$. This can be rewritten $(2\delta)/c_{t+1} = 1/c_t$ or $c_{t+1} = (2\delta)c_t$. Thus $c_t = (2\delta)^t c_0$. It follows that the budget constraint is $p_0 x = \sum_{t=0}^{\infty} \delta^t p_0 c_0 = p_0 c_0/(1-\delta)$, or $c_0 = (1-\delta)x$. We may normalize prices so $p_0 = 1$.

   b) The Malinvaud prices were found in part (a). They are $p_t = 2^{-t}$.

   c) Since $c_0 = (1-\delta)x$, $a_0 = \delta x$. Then $f(a_0) = 2\delta x$. Since $c_1 = 2\delta c_0$, $a_1 = 2\delta^2 x$. By induction, $a_t = 2^t \delta^{t+1} x$ and $p_t a_t = \delta^{t+1} \to 0$, establishing the transversality condition.

4. Consider a production economy with 2 goods and 2 consumers. There is one firm with technology set $Y = \{(y_1, y_2) : 2y_2 \le -y_1, \text{and } 2y_1 \le -y_2\}$. Both consumers have equal-weighted Cobb-Douglas utility: $u_i = (x^i_1 x^i_2)^{1/2}$. Endowments are $\boldsymbol{\omega}^1 = (1, 2)$ and $\boldsymbol{\omega}^2 = (0, 3)$.

   a) Find all equilibrium allocations.

   b) How the equilibrium change if the endowments were $\boldsymbol{\omega}^1 = (2, 1)$ and $\boldsymbol{\omega}^2 = (3, 0)$?

   Answer:

   a) Since utility is Cobb-Douglas, we know that no price can be zero. We normalize prices so $\mathbf{p} = (1, p)$. The constant returns to scale production means that profit is always zero, so the only income is from the endowments. The incomes are $m^1 = 1 + 2p$ and $m^2 = 3p$. Then consumer demands are $\mathbf{x}^1(p) = (1 + 2p)(1/2, 1/2p)$ and $\mathbf{x}^2(p) = 3p(1/2, 1/2p)$ so consumer demand is $(1 + 5p)(1/2, 1/2p)$. The aggregate endowment is $\boldsymbol{\omega} = (1, 5)$.

   We first check if there is an equilibrium without production. It would require $(1 + 5p)/2 = 1$, or $p = 1/5$. But at that price, it is optimal to use good 2 to produce good one. If we produce good one, the price must be $p = 1/2$. Then consumer demand is $(7/4, 7/2)$. With an endowment of $(1, 5)$, this requires production vector $\mathbf{y} = (3/4, -3/2)$, which is feasible and yields zero profit when $\mathbf{p} = (1, 1/2)$. (Note that producing good two yields negative profit.) The resulting allocation of consumption is $\mathbf{x}^1 = (1, 2)$ and $\mathbf{x}^2 = (3/4, 3/2)$.

   b) Here the incomes are $2 + p$ and 3, so consumer demand is $(5 + p)(1/2, 1/2p)$. The aggregate endowment is $\boldsymbol{\omega} = (5, 1)$. Without production, the equilibrium price would be $p = 5$. At that price, it is profitable to production good 2 from good 1. But if we produce good 2, the price must be $p = 2$. In that case market demand is $(7/2, 7/4)$, which requires production vector $\mathbf{y} = (-3/2, 3/4)$. This is feasible and profit-maximizing and gives us the equilirium. Thus prices are $\mathbf{p} = (1, 2)$, production is $\mathbf{y} = (-3/2, 3/4)$ and consumption is $\mathbf{x}^1 = (2, 1)$ and $\mathbf{x}^2 = (3/2, 3/4)$.

5. Consider an exchange economy with 2 consumers, 1 good, and 3 states of the world. Let x^i_s denote consumer i's

consumption of the one good in state s. Each consumer has utility function

$$u(\mathbf{x}^i) = \sum_{s=1}^{3} \ln x_s^i.$$

The endowments are $\omega^1 = (1,2,3)$ and $\omega^2 = (3,2,1)$. Find the Arrovian securities equilibrium. Be sure to indicate the spot market prices, prices of the securities, and the equilibrium consumption by each consumer.

Hint: Since we can normalize each spot market separately, we can set $p_s = 1$ for each s.

Answer: With $p_s = 1$ for every s, both consumers consume their income in state s. I.e., $x_{1s}^i = \omega_{1s}^i + z_s^i$. Thus indirect utility is $v^i(z) = \sum_{s=1}^3 \ln(\omega_{1s}^i + z_s^i)$. The first-order conditions are then $\lambda q_s = 1/(\omega_{1s}^i + z_s^i)$ or $\omega_{1s}^i + z_s^i = 1/(\lambda q_s)$. Summing over i and using asset market clearing, we find $4 = 2/(\lambda q_s)$. The q_s must be equal, so we may set $q_s = 1$ for every security s. Now $\lambda = 1/2$. Substituting back in the first-order conditions, we find $2 = \omega_{1s}^i = z_s^i$. Thus $z^1 = (1,0,-1)$ and $z^2 = (-1,0,1)$. The corresponding goods allocations are $\mathbf{x}^1 = (2,2,2)$ and $\mathbf{x}^2 = (2,2,2)$.