1. A consumer has utility \( u(x, y) = \sqrt{x} + \sqrt{y} \). Prices are given by the vector \( p \in \mathbb{R}^2_{++} \) and income is \( m > 0 \).

\( a) \) Find the indirect utility function.

\( b) \) Find the expenditure function.

**Answer:**

\( a) \) Note that the marginal utility is infinite when either \( x = 0 \) or \( y = 0 \). This means the solution will be interior. Form \( L = \sqrt{x} + \sqrt{y} - \lambda(p_x x + p_y y - m) \). The first-order conditions are \( \lambda p_x = 1/2\sqrt{x} \) and \( \lambda p_y = 1/2\sqrt{y} \). Eliminating \( \lambda \), we find \( y/x = p_x^2/p_y^2 \). Substituting in the budget constraint yields \( x = mp_y/(p_x^2 + p_x p_y) \) and \( y = mp_x/(p_y^2 + p_x p_y) \). It follows that

\[
\nu(p, m) = (1 + p_x) \sqrt{\frac{mp_y}{p_x^2 + p_x p_y}} = \sqrt{\frac{m(p_x + p_y)}{p_x p_y}}.
\]

\( b) \) We now use the duality relation \( \nu(p, e(p, \bar{u})) = \bar{u} \) to find

\[
e(p, \bar{u}) = \bar{u} \frac{p_x p_y}{p_x + p_y}.
\]

2. Consider a two-person, two-good exchange economy. Consumer 1 has utility \( u_1(x) = \min[x_1, x_2] \) and consumer 2 has utility \( u_2(x) = \sqrt{x_1 x_2} \). The endowments are \( \omega^1 = (3, 1) \) and \( \omega^2 = (1, 2) \).

\( a) \) Find all the equilibria of this economy.

\( b) \) Find all Pareto optima.

\( c) \) Find the core.

**Answer:**

\( a) \) Since consumer 2 has Cobb-Douglas preferences, we know that both goods will have positive prices in equilibrium. Take good one as the numéraire and set \( p_2 = p \). Since both goods have positive prices, consumer 1 will not waste any income and \( x^1_1 = x^2_1 \). Since \( m_1 = (1, p) - \omega^1 = 3 + p \), the budget constraint is \( x^1_1 + px^2_1 = 3 + p \). It follows that \( x^1(p) = (3 + p)/(1 + p)(1, 1) \). As \( m_2 = 1 + 2p \) and 2’s utility is equal-weighted Cobb-Douglas, \( x^2(p) = (1 + 2p)/(1, 1/p) \).

Demand equals supply for good 1, so \( (3 + p)/(1 + p) + (1 + 2p)/2 = \omega_1 = 4 \). Equivalently, \( 6 + 2p + 1 + 3p + p^2 = 8 + 8p \) implying \( 2p^2 - 3p - 1 = 0 \). This has only one positive solution, \( p = (3 + \sqrt{17})/4 \).

The demand functions then yield the allocation of goods.

\( b) \) Here \( \text{MRS}^2 = x_2^2/x_1^2 \). It is clear that the Leontief corner points for consumer 1 are all Pareto optimal. Once you hit the upper boundary, the Pareto set follows the upper boundary to the corner. It is \( \{(x_1, x_2) : (x_1 = x_2 \text{ and } 0 \leq x_1 \leq 3) \text{ or } (x_2 = 3 \text{ and } 3 \leq x_1 \leq 4)\} \).

\( c) \) Besides being Pareto optimal, the core points must be individually rational. This requires \( x_1 \geq 1 \) (so \( u_1 \geq u_1(3, 1) = 1 \)) and \( x_1 \leq 2 \) (for \( u_2 \geq u_2(2, 1) = \sqrt{2} \)). The latter follows from the equation \( \sqrt{(4 - x_1)(3 - x_1)} \geq \sqrt{2} \).

3. There is a single consumption good available at each point in time. An infinitely-lived consumer has period utility \( u(c) = \sqrt{c} \) and discount factor \( \delta, 0 < \delta < 1 \). The consumer has discounted wealth \( W = 1 \). Prices at time \( t = 0 \) are given by \( p_t = (1 + r)^{-t} \). Find the optimal consumption path. What restrictions do you need on \( \delta \) and \( r \)?
4. Suppose a firm’s production set is given by $Y$. Form the Lagrangian $\mathcal{L} = U(c) - \lambda \left( \sum_{t=0}^{\infty} p_t c_t - 1 \right)$. The first-order conditions are $\delta \sqrt{c_t} = \lambda p_t$. Dividing to eliminate $\lambda$ yields

$$\frac{p_t}{p_{t+1}} = \delta \sqrt{\frac{c_{t+1}}{c_t}}$$

Setting $\beta = [\delta(1+r)]^2$, we obtain $c_t = \beta^t c_0$. The budget constraint is then

$$1 = c_0 \sum_{t=0}^{\infty} \frac{\beta^t}{(1+r)^t} = c_0 \sum_{t=0}^{\infty} [(1+r)\delta^2]^t.$$ 

This converges if $(1+r)\delta^2 < 1$ (the required condition) in which case $c_0$ is the reciprocal of the sum, $c_0 = 1 - (1-r)\delta^2$.

4. Suppose a firm’s production set is given by $Y = ((-z, q) : z \geq 0, q \leq z^{1/3})$.

a) Find the profit-maximizing net output vector.

b) Derive the profit function $\pi(p_z, p_q)$.

c) Does the technology exhibit constant returns to scale? Increasing returns to scale? Decreasing returns to scale?

Answer:

a) Profit is $p_q z^{1/3} - p_z z$. The first-order condition for profit maximization is $p_q z^{-2/3} / 3 = p_z$, so $z = (p_q/3p_z)^{3/2}$ and $q = (p_q/3p_z)^{1/2}$. The net output is $(-p_q/3p_z)^{3/2}, (p_q/3p_z)^{1/2})$.

b) The maximum profit obtained is then $p_q^3/3 (3p_z)^{-1/2} - p_z^3/3 (3p_z)^{-1/2} = 2p_q^3/3 p_z^{-1/2}/3^{3/2}$.

c) The production function is strictly concave, so there are decreasing returns to scale.

5. Consider an exchange economy with 2 consumers, 2 goods, and 2 states of the world. Let $x_{is}$ denote consumer $i$’s consumption of good $\ell$ in state $s$. Each consumer has utility function

$$u(x^i) = \sum_{i,s=1}^{2} \frac{1}{4} \ln x_{is}^i.$$ 

The endowments are $\omega^1 = ((1,2),(1,3))$ and $\omega^2 = ((2,1),(3,2))$. Find the spot prices and securities prices for the Arrowian securities equilibrium.

Answer: We consider the spot markets first. Once again, we have Cobb-Douglas preferences and the equilibrium prices will be positive. We choose good 1 as the numéraire in each state. Incomes in each state are then $m_1^1 = 1 + 2p_{21} + z_1^1$, $m_1^2 = 1 + 3p_{22} + z_1^2$, $m_2^1 = 2 + p_{21} + z_2^1$, $m_2^2 = 3 + 2p_{22} + z_2^2$. Since markets clear, $z_1^1 + z_1^2 = 0$ and $z_1^2 + z_2^2 = 0$, so aggregate income in each state is $m_1 = 3 + 3p_{21}$ and $m_2 = 4 + 5p_{22}$. Equal-weighted Cobb-Douglas preferences yield aggregate demand of $x_1(p_1) = (m_1/2)(1,1/p_{21})$ and $x_2(p_2) = (m_2/2)(1,1/p_{22})$. To clear market 1 in each state we must have $3 = (3 + 3p_{21})/2$ and $4 = (4 + 5p_{22})/2$. Thus $p_{21} = 1$ and $p_{22} = 4/5$. The equilibria in the spot markets yield $x_1^1 = (3 + z_1^1)/2(1,1), x_2^1 = (3 + 3z_1^1)/2(1,1), x_1^2 = (17 + 5z_2^1)/10(1,5/4), x_2^2 = (23 + 5z_2^2)/10(1,5/4)$.

Indirect utility at time 0 is now

$$v_1(z^1) = \frac{1}{2} \ln \frac{3 + z_1^1}{2} + \frac{1}{2} \ln \frac{17 + 5z_1^1}{10} + \frac{1}{4} \ln \frac{5}{4}$$

and

$$v_2(z^2) = \frac{1}{2} \ln \frac{3 + z_2^1}{2} + \frac{1}{2} \ln \frac{23 + 5z_2^2}{10} + \frac{1}{4} \ln \frac{5}{4}.$$
We solve for the securities equilibrium by maximizing indirect utility under the budget constraint $z^1 \cdot q = 0$. Let \( q = (1, q) \). Then $z_1^1 = -qz_2^1$ and we can maximize

$$
\ln \frac{3 - qz_2^1}{2} + \ln \frac{17 + 5z_1^1}{10}
$$

and

$$
\ln \frac{3 - qz_2^2}{2} + \ln \frac{23 + 5z_2^2}{10}.
$$

Then

$$
q \frac{2}{3 - qz_2^1} = \frac{10}{17 + 5z_2^1} \quad \text{and} \quad q \frac{2}{3 - qz_2^2} = \frac{10}{23 + 5z_2^2}.
$$

Solving for the $z_2^1$, we obtain

$$
z_2^1 = \frac{15 - 17q}{10q} \quad \text{and} \quad z_2^2 = \frac{15 - 23q}{10q}.
$$

Market clearing requires $z_1^2 + z_2^2 = 0$. Thus $30 - 40q = 0$ or $q = 3/4$. Then $z_1^2 = 3/10$ and $z_2^2 = -3/10$. One can substitute back to obtain the allocations.