Micro II Final, December 11, 2013

1. A consumer has utility $u(x, y) = \sqrt{x} + \sqrt{y}$. Prices are given by the vector $\mathbf{p} \in \mathbb{R}^2_{++}$ and income is $m > 0$.

   a) Find the indirect utility function.

   b) Find the expenditure function.

   Answer:

   a) Note that the marginal utility is infinite when either $x = 0$ or $y = 0$. This means the solution will be interior. Form $\mathcal{L} = \sqrt{x} + \sqrt{y} - \lambda(p_x x + p_y y - m)$. The first-order conditions are $\lambda p = 1/2\sqrt{x}$ and $\lambda p_y = 1/2\sqrt{y}$. Eliminating λ, we find $y/x = p_x^2/p_y^2$. Substituting in the budget constraint yields $x = mp_y/(p_x^2 + p_x p_y)$ and $y = mp_x/(p_y^2 + p_x p_y)$. It follows that

$$v(\mathbf{p}, m) = (1 + \frac{p_x}{p_y})\sqrt{\frac{mp_y}{p_x^2 + p_x p_y}} = \sqrt{\frac{m(p_x + p_y)}{p_x p_y}}.$$

   b) We now use the duality relation $v(\mathbf{p}, e(\mathbf{p}, \bar{u})) = \bar{u}$ to find

$$e(\mathbf{p}, \bar{u}) = \bar{u}^2 \frac{p_x p_y}{p_x + p_y}.$$

2. Consider a two-person, two-good exchange economy. Consumer 1 has utility $u_1(\mathbf{x}) = \min\{x_1, x_2\}$ and consumer 2 has utility $u_2(\mathbf{x}) = \sqrt{x_1 x_2}$. The endowments are $\boldsymbol{\omega}^1 = (3, 1)$ and $\boldsymbol{\omega}^2 = (1, 2)$.

   a) Find all the equilibria of this economy.

   b) Find all Pareto optima.

   c) Find the core.

   Answer:

   a) Since consumer 2 has Cobb-Douglas preferences, we know that both goods will have positive prices in equilibrium. Take good one as the numéraire and set $p_2 = p$. Since both goods have positive prices, consumer 1 will not waste any income and $x_1^1 = x_2^1$. Since $m_1 = (1, p) \cdot \boldsymbol{\omega}^1 = 3 + p$, the budget constraint is $x_1^1 + px_2^1 = 3 + p$. It follows that $\mathbf{x}^1(\mathbf{p}) = (3 + p)/(1 + p)(1, 1)$. As $m_2 = 1 + 2p$ and 2's utility is equal-weighted Cobb-Douglas, $\mathbf{x}^2(\mathbf{p}) = (1 + 2p)/2(1, 1/p)$.

   Demand equals supply for good 1, so $(3 + p)/(1 + p) + (1 + 2p)/2 = \omega_1 = 4$. Equivalently, $6 + 2p + 1 + 3p + p^2 = 8 + 8p$, implying $2p^2 - 3p - 1 = 0$. This has only one positive solution, $p = (3 + \sqrt{17})/4$. The demand functions then yield the allocation of goods.

   b) Here $\text{MRS}^2 = x_2^2/x_1^2$. It is clear that the Leontief corner points for consumer 1 are all Pareto optimal. Once you hit the upper boundary, the Pareto set follows the upper boundary to the corner. It is $\{(x_1, x_2) : (x_1 = x_2 \text{ and } 0 \le x_1 \le 3) \text{ or } (x_2 = 3 \text{ and } 3 \le x^1 \le 4)\}$.

   c) Besides being Pareto optimal, the core points must be individually rational. This requires $x_1 \ge 1$ (so $u_1 \ge u_1(3, 1) = 1$) and $x_1 \le 2$ (for $u_2 \ge u_2(2, 1) = \sqrt{2}$). The latter follows from the equation $\sqrt{(4 - x_1)(3 - x_1)} \ge \sqrt{2}$.

3. There is a single consumption good available at each point in time. An infinitely-lived consumer has period utility $u(c) = \sqrt{c}$ and discount factor δ, $0 < \delta < 1$. The consumer has discounted wealth $W = 1$. Prices at time $t = 0, \ldots$ are given by $p_t = (1 + r)^{-t}$. Find the optimal consumption path. What restrictions do you need on δ and r?

Answer: The objective is $U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t \sqrt{c_t}$. Note that the infinite marginal utility at 0 implies the only solutions will obey $c_t > 0$ for all t. Form the Lagrangian $\mathcal{L} = U(\mathbf{c}) - \lambda(\sum_{t=0}^{\infty} p_t c_t - 1)$. The first-order conditions are $\delta^t / 2\sqrt{c_t} = \lambda p_t$. Dividing to eliminate λ yields

$$\frac{p_t}{p_{t+1}} = \delta \sqrt{\frac{c_{t+1}}{c_t}}.$$

Setting $\beta = [\delta(1+r)]^2$, we obtain $c_t = \beta^t c_0$. The budget constraint is then

$$1 = c_0 \sum_{t=0}^{\infty} \frac{\beta^t}{(1+r)^t} = c_0 \sum_{t=0}^{\infty} [(1+r)\delta^2]^t.$$

This converges if $(1+r)\delta^2 < 1$ (the required condition) in which case c_0 is the reciprocal of the sum, $c_0 = 1 - (1-r)\delta^2$.

4. Suppose a firm's production set is given by $Y = \{(-z, q) : z \geq 0, q \leq z^{1/3}\}$.

   a) Find the profit-maximizing net output vector.

   b) Derive the profit function $\pi(p_z, p_q)$.

   c) Does the technology exhibit constant returns to scale? Increasing returns to scale? Decreasing returns to scale?

Answer:

a) Profit is $p_q z^{1/3} - p_z z$. The first-order condition for profit maximization is $p_q z^{-2/3}/3 = p_z$, so $z = (p_q/3p_z)^{3/2}$ and $q = (p_q/3p_z)^{1/2}$. The net output is $(-(p_q/3p_z)^{3/2}, (p_q/3p_z)^{1/2})$.

b) The maximum profit obtained is then $p_q^{3/2}(3p_z)^{-1/2} - p_z^{3/2} p^{-1/2} 3^{-3/2} = 2p_q^{3/2} p_z^{-1/2}/3^{3/2}$.

c) The production function is strictly concave, so there are decreasing returns to scale.

5. Consider an exchange economy with 2 consumers, 2 goods, and 2 states of the world. Let $x_{\ell s}^i$ denote consumer i's consumption of good ℓ in state s. Each consumer has utility function

$$u(\mathbf{x}^i) = \sum_{\ell,s=1}^{2} \frac{1}{4} \ln x_{\ell s}^i.$$

The endowments are $\boldsymbol{\omega}^1 = \big((1,2),(1,3)\big)$ and $\boldsymbol{\omega}^2 = \big((2,1),(3,2)\big)$. Find the spot prices and securities prices for the Arrovian securities equilibrium.

Answer: We consider the spot markets first. Once again, we have Cobb-Douglas preferences and the equilibrium prices will be positive. We choose good 1 as the numéraire in each state. Incomes in each state are then $m_1^1 = 1 + 2p_{21} + z_1^1$, $m_2^1 = 1 + 3p_{22} + z_2^1$, $m_1^2 = 2 + p_{21} + z_1^2$, $m_2^2 = 3 + 2p_{22} + z_2^2$. Since markets clear, $z_1^1 + z_1^2 = 0$ and $z_2^1 + z_2^2 = 0$, so aggregate income in each state is $m_1 = 3 + 3p_{21}$ and $m_2 = 4 + 5p_{22}$. Equal-weighted Cobb-Douglas preferences yield aggregate demand of $\mathbf{x}_1(\mathbf{p}_1) = (m_1/2)(1, 1/p_{21})$ and $\mathbf{x}_2(\mathbf{p}_2) = (m_2/2)(1, 1/p_{22})$. To clear market 1 in each state we must have $3 = (3 + 3p_{21})/2$ and $4 = (4 + 5p_{22})/2$. Thus $p_{21} = 1$ and $p_{22} = 4/5$. The equilibria in the spot markets yield $\mathbf{x}_1^1 = (3 + z_1^1)/2(1,1)$, $\mathbf{x}_1^2 = (3 + 3z_1^2)/2(1,1)$, $\mathbf{x}_2^1 = (17 + 5z_2^1)/10(1, 5/4)$, $\mathbf{x}_2^2 = (23 + 5z_2^2)/10(1, 5/4)$.

Indirect utility at time 0 is now

$$v_1(z^1) = \frac{1}{2} \ln \frac{3 + z_1^1}{2} + \frac{1}{2} \ln \frac{17 + 5z_2^1}{10} + \frac{1}{4} \ln \frac{5}{4}$$

and

$$v_2(z^2) = \frac{1}{2} \ln \frac{3 + z_1^2}{2} + \frac{1}{2} \ln \frac{23 + 5z_2^2}{10} + \frac{1}{4} \ln \frac{5}{4}.$$

We solve for the securities equilibrium by maximizing indirect utility under the budget constraint $\mathbf{z}^i \cdot \mathbf{q} = 0$. Let $\mathbf{q} = (1, q)$. Then $z_1^i = -qz_2^i$ and we can maximize

$$\ln \frac{3 - qz_2^1}{2} + \ln \frac{17 + 5z_2^1}{10}$$

and

$$\ln \frac{3 - qz_2^2}{2} + \ln \frac{23 + 5z_2^2}{10}.$$

Then

$$q \frac{2}{3 - qz_2^1} = \frac{10}{17 + 5z_2^1} \quad \text{and} \quad q \frac{2}{3 - qz_2^2} = \frac{10}{23 + 5z_2^2}.$$

Solving for the z_2^i, we obtain

$$z_2^1 = \frac{15 - 17q}{10q} \quad \text{and} \quad z_2^2 = \frac{15 - 23q}{10q}.$$

Market clearing requires $z_2^1 + z_2^2 = 0$. Thus $30 - 40q = 0$ or $q = 3/4$. Then $z_1^2 = 3/10$ and $z_2^2 = -3/10$. One can substitute back to obtain the allocations.