Micro II Midterm, October 13, 2015

1. Suppose a utility function u has indirect utility $v(\mathbf{p}, m) = a(\mathbf{p}) + mb(\mathbf{p})$ for some functions a and b. Let φ be an increasing differentiable function.

   a) What is the indirect utility function of $w = \varphi \circ u$?

   Answer: Since $v(\mathbf{p}, m) = u\big(x(\mathbf{p}, m)\big)$, we simply replace u by w, obtaining $\varphi\big(a(\mathbf{p}) + mb(\mathbf{p})\big)$ as the indirect utility function.

   b) What is the expenditure function for w?

   Answer: We use duality: $\bar{u} = \varphi\big(a(\mathbf{p}) + e(\mathbf{p}, \bar{u})b(\mathbf{p})\big)$, so $\varphi^{-1}(\bar{u}) = a(\mathbf{p}) + e(\mathbf{p}, \bar{u})b(\mathbf{p})$. Solving for e, we find $e(\mathbf{p}, \bar{u}) = -a(\mathbf{p})/b(\mathbf{p}) + \varphi^{-1}(\bar{u})/b(\mathbf{p})$.

   c) Suppose consumers $i = 1, \ldots, I$ have expenditure functions given by $e_i(\mathbf{p}, \bar{u}) = i^2 p_1 + (\sum p_\ell)/\sqrt{\bar{u}}$. If we aggregate the resulting Marshallian demand functions, is the result independent of the distribution of income? Why or why not?

   Answer: Unfortunately, I didn't ask what I meant to. The expenditure functions should have been $e_i(\mathbf{p}, \bar{u}) = i^2 p_1 + (\sum p_\ell)\sqrt{\bar{u}}$. By part (b), they are equivalent to expenditure functions of the form $i^2 p_1 + \bar{u}(\sum p_\ell)$, which are in Gorman form. Because they are in Gorman form, and the $\bar{u}$ term is identical, they can be aggregated without concern about the distribution of income.

   As writen, we cannot use parts (a) and (b). Instead, we must derive the indirect utility function and use Roy's Idenity to compute the Mashallian demand.

   Thus $m_i = i^2 p_1 + (\sum p_\ell)/\sqrt{v_i}$, so $\sqrt{v_i}(m_i - i^2 p_1) = (\sum p_\ell)$, yielding $\sqrt{v_i} = (\sum p_\ell)/(m_i - i^2 p_1)$. So $v_i = (\sum p_\ell)^2/(m_i - i^2 p_1)^2$. Note that this is not a proper indirect utility function because it is decreasing in income (both goods are bads). If it was a legitimate indirect utility function, we would do the following.

   Taking the square root of utility, we find this is equivalent to $V_i = (\sum p_\ell)/(m_i - i^2 p_1)$.

   Then $\partial V_i/\partial m_i = -(\sum p_\ell)/(m_i - i^2 p_1)^2$ and

$$\frac{\partial V_i}{\partial p_k} = \begin{cases} \frac{1}{(m_i - i^2 p_1)} + \frac{i^2(\sum p_\ell)}{(m_i - i^2 p_1)^2} & \text{if } k = 1 \\ \frac{1}{(m_i - i^2 p_1)} & \text{if } k \neq 1. \end{cases}$$

   It follows that

$$x_k^i = -\frac{\partial V_i/\partial p_k}{\partial V_i/\partial m_i} = \begin{cases} m_i - i^2 p_1 - i^2 & \text{if } k = 1 \\ m_i - i^2 p_1 & \text{if } k \neq 1. \end{cases}$$

We add to find aggregate demand. It is

$$x_k = -\frac{\partial V_i/\partial p_k}{\partial V_i/\partial m_i} = \begin{cases} m - (p_1 + 1)\sum_i i^2 & \text{if } k = 1 \\ m - p_1 \sum_i i^2 & \text{if } k \neq 1. \end{cases}$$

2. There are two goods. Suppose a consumer has the utility function $u(x_1, x_2) = \sqrt{x_1} + 2x_2$ and consumption set $\mathfrak{X} = \mathbb{R}^2_+$. Let prices be $\mathbf{p} = (1, p) \gg 0$ and income be $m > 0$.

   a) Find the ordinary (Marshallian) demand $\mathbf{x}(\mathbf{p}, m)$.

   Answer: We form the Lagrangian $\mathcal{L} = \sqrt{x_1} + 2x_2 - \lambda(x_1 + px_2) + \mu_1 x_1 + \mu_2 x_2$. The first-order conditions are

$$\lambda = \frac{1}{2\sqrt{x_1}} + \mu_1 \tag{1}$$

$$\lambda p = 2 + \mu_2. \tag{2}$$

   Note $\lambda p \geq 2$, so the budget constraint binds by complementary slackness. Also, if $x_1 = 0$, equation (1) cannot be satisfied. Thus $x_1 > 0$, implying that $\mu_1 = 0$ by complementary slackness.

   In case I, $x_2 > 0$. Then $\mu_2 = 0$ by complementary slackness. Then $\lambda p = 2$ by equation (2), so $1/2\sqrt{x_1} = 2$ by equation (1). This yields $x_1 = p^2/16$. For this to be feasible, we must have $m \geq p^2/16$. In that case $x_2 = (m - p^2/16)/p$.

   In case II, $x_2 = 0$. Then $\lambda p \geq 2$ by equation (2) and $x_1 = m$ from the budget constraint. This requires that $m = x_1 \leq p^2/16$.

   Thus Marshallian demand is

$$\mathbf{x}(\mathbf{p}, m) = \begin{cases} (m, 0) & \text{if } m \leq p^2/16 \\ (\frac{p^2}{16}, p^{-1}(m - \frac{p^2}{16})) & \text{if } m \geq p^2/16. \end{cases}$$

   b) Compute the indirect utility function $v(\mathbf{p}, m)$.

   Answer: Indirect utility is

$$v(p, m) = \begin{cases} \sqrt{m} & \text{if } m \leq 1/16 \\ \frac{1}{4} + p^{-1}(2m - \frac{1}{8}) & \text{if } m \geq 1/16. \end{cases}$$

3. Suppose a firm has production function $f(z) = z^\alpha$ where $\alpha > 0$ and faces output price $p > 0$ and input price $w > 0$.

a) For what values of α is it possible to maximize profit?

Answer: Profit is $\pi(z) = pz^{\alpha} - wz$. Marginal profit is $\pi' = \alpha p z^{\alpha-1} - w$. If $\alpha > 1$, marginal profit will be positive for large values of z. In this case profit cannot be maximized. If $\alpha < 1$, marginal profit will eventually become negative. This means we are working on a compact set, and profit can be maximized. Finally, if $\alpha = 1$, $\pi' = p - w$. If $p > w$, it is not possible to maximize profit, while if $p \leq w$, maximized profit is zero.

b) For the values of α where profit can be maximized, compute the supply function $y(p, w)$.

Answer: When $\alpha < 1$, the first-order condition is $p\alpha z^{\alpha-1} = w$. Solving for z, we find $z = (w/p\alpha)^{1/(\alpha-1)}$. Since supply is z^{α}, supply is $y(p, w) = (w/p\alpha)^{\alpha/(\alpha-1)}$.

When $\alpha = 1$, profit can be maximized if $p \leq w$. If $p < w$, marginal profit is always negative, so $z = 0$ and supply $y(p, w) = 0$. If $p = w$, any level of $z \geq 0$ will maximize profit and the supply correspondence is $y(p, w) = \{y : y \geq 0\}$.

4. A consumer has utility function given by $U(x) = \sum_{\ell=1}^{L} u(x_\ell)$ for some C^2 function u with $u' > 0$ and $u'' < 0$. Does the demand for good ℓ obey the Law of Demand? Why or why not?

Answer: The demand for good ℓ does obey the Law of Demand. As we saw in Problem 2.13, all goods are normal goods when utility is additively separable. This means that the income elasticity is positive. The Slutsky equation now says that $\partial x_\ell/\partial p_\ell = \partial h_\ell/\partial p_\ell - x_\ell \partial x_\ell/\partial m$ where h denotes Hicksian demand. Because $D_p h$ is negative semi-definite, $\partial h_\ell/\partial p_\ell \leq 0$. Normality implies $\partial x_\ell/\partial m > 0$. This means $\partial x_\ell/\partial p_\ell < 0$, the demand curve has negative slope.