Micro II Final, May 4, 2016

1. The expenditure function is $e(\mathbf{p},\bar{u}) = 2\bar{u}^{1/20}\sqrt{p_1p_2}$.

   a) Find the indirect utility function.

   b) Find the Marshallian demand functions.

   Answer:

   a) We use duality to find the indirect utility function. We know that $e\big(\mathbf{p}, v(\mathbf{p}, m)\big) = m$. Thus $2v^{1/20}(p_1p_2)^{1/2} = m$. It follows that

   $$v(\mathbf{p}, m) = \frac{m^{20}}{2^{20}(p_1p_2)^{10}}.$$

   b) We use Roy's identity to find the demand functions. Here $\partial v/\partial m = 20m^{19}2^{-20}/(p_1p_2)^{10}$ and $\partial v/\partial p_i = -10m^{20}/p_i^{11}p_j^{10}$ where $i \neq j$. By Roy's identity, $x_i(\mathbf{p}, m) = m/2p_i$. As you might guess, the utility function is equivalent to equal-weighted Cobb-Douglas.

2. An economy has two goods. There is one firm which uses good one as input to produce good two. Its production function is $f(z) = 3z$. There are two Cobb-Douglas consumers, both with $u(x_1, x_2) = x_1x_2$. Endowments are $\boldsymbol{\omega}^1 = (3, 0)$, $\boldsymbol{\omega}^2 = (2, 1)$. Find all Walrasian equilibria. Does the firm produce anything in equilibrium?

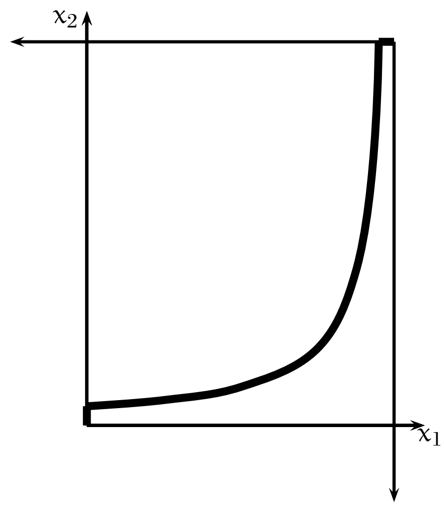
   Answer: The firm's profit is $3p_2z - p_1z$ where z is its input of good one. If $z > 0$, profit maximization requires that $3p_2 = p_1$. Since preferences are Cobb-Douglas, neither price is zero. We can take good one as numéraire, yielding equilibrium price vector $\mathbf{p} = (1, 1/3)$. Since maximum profit is always zero, income is $m_1 = 3$ and $m_2 = 7/3$. With this Cobb-Douglas utility $p_1x_1^i = p_2x_2^i = m_i/2$. It follows that $\mathbf{x}^1 = (3/2, 9/2)$ and $\mathbf{x}^2 = (7/6, 7/2)$. The total endowment is $(5, 4)$ We subtract consumer demand for good one to find the firm's input, $z = 5 - 3/2 - 7/6 = 7/3$. The firm produces 7 units of good two. It's production vector is $\mathbf{y} = (-7/3, 7)$. This means that the supply of good two is 8, which is precisely equal to demand.

   There is no Walrasian equilibrium where the firm doesn't produce. Since the preferences are identical Cobb-Douglas, both consumer would get a share of the total endowment. Then $MRS_{12} = 5/1 = p_1/p_2$, so $p_2 = 5p_1$. At those prices, increasing production increases profit, so it cannot be an equilibrium.

3. An economy has two goods. Consumer one has utility $u_1(\mathbf{x}^1) = x_1^1 + 2\sqrt{x_2^1}$ while consumer two has utility $u_2(\mathbf{x}^2) = 2\sqrt{x_1^2} + x_2^2$. The total endowment is $\boldsymbol{\omega} = (4, 5)$.

   Find all Pareto optimal allocations of goods.

Answer: We first look for interior Pareto optima. The marginal rates of substitution are $MRS^1_{12} = 2\sqrt{x^1_2}$ and $MRS^2_{12} = 1/\sqrt{x^2_1}$. At any interior Pareto optimum, $MRS^1_{12} = MRS^2_{12}$. It follows that $x^1_2 x^2_1 = 1$. Since allocations obey $x^1_1 + x^2_1 = 4$ and $x^1_2 + x^2_2 = 5$, we can write the Pareto condition as $x^1_2(4 - x^1_1) = 1$. This is a rectangular hyperbola pointing left and asymptotic to the horizontal axis and the line $x^1_1 = 4$. It intercepts the left boundary of the Edgeworth box when $x^1_1 = 0$ and so $x^1_2 = 1/4$. It intercepts the top boundary when $x^1_2 = 5$ yielding $x^1_1 = 3.8$. Moving along the top boundary from $(3.8, 5)$ to $(4, 5)$ keeps MRS^1_{12} constant while increasing MRS^2_{12}, which is precisely what we need to have the boundary points be Pareto optimal. Similarly, the boundary segment from $(0, 0)$ to $(0, 1/4)$ is also Pareto optimal. This illustrated in the diagram.



4. Suppose a consumer has intertemporal utility function $U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(c_t)$ where c_t is the amount of the one consumption good available in period t (i.e., $L = 1$) and $0 < \delta < 1$. Felicity obeys $u' > 0$ and $u'' < 0$. Let a price sequence p_t be given. The budget constraint is $\sum_{t=0}^{\infty} p_t c_t = W$.

   a) What are the first-order conditions for an interior optimum in terms of the marginal felicity. Make sure you have eliminated any Lagrange multipliers from the equations.

   b) Suppose W changes, holding everything else constant. Use the first-order conditions to express $\partial c_{t+1}/\partial W$ in terms of $\partial c_t/\partial W$. How do their signs relate?

   c) Use the budget constraint and the result from (b) to determine the sign of $\partial c_t/\partial W$ for every t.

Answer:

a) The Lagrangian is $\mathcal{L} = U(\mathbf{c}) - \lambda(\sum_t p_t c_t - W)$, The first-order conditions are $\delta^t u'(c_t) = \partial U/\partial c_t = \lambda p_t$. We can eliminate λ by dividing the $t + 1$ equation by the time t equation.

This yields

$$\frac{\delta u'(c_{t+1})}{u'(c_t} = \frac{p_{t+1}}{p_t}.$$

b) We rewrite the first-order conditions as $\delta u'(c_{t+1}) = (p_{t+1}/p_t)u'(c_t$. Differentiating with respect to W yields $\delta u''\partial c_{t+1}/\partial W = u''\partial c_t/\partial W$. Since both u'' terms are negative, $\partial c_t/\partial W$ and $\partial c_{t+1}/\partial W$ have the same sign

c) Since the budget constraint must be satisfied, at least one c_τ must increase when W increases. As all $\partial c_t/\partial W$ have the the same sign, all must be positive.

5. Consider an exchange economy with 2 consumers, 1 good, and 3 states of the world. Let x_s^i denote consumer i's consumption of the one good in state s. Each consumer has utility function

$$u(\mathbf{x}^i) = \sum_{s=1}^{3} \ln x_s^i.$$

The endowments are $\boldsymbol{\omega}^1 = (1,2,3)$ and $\boldsymbol{\omega}^2 = (3,2,1)$. Find the Arrovian securities equilibrium. Be sure to indicate the spot market prices, prices of the securities, and the equilibrium consumption by each consumer.

Hint: Since we can normalize each spot market separately, we can set $p_s = 1$ for each s.

Answer: With $p_s = 1$ for every s, both consumers consume their income in state s. I.e., $x_{1s}^i = \omega_{1s}^i + z_s^i$. Thus indirect utility is $v^i(\mathbf{z}) = \sum_{s=1}^{3} \ln(\omega_{1s}^i + z_s^i)$. The first-order conditions are then $\lambda_i q_s = 1/(\omega_{1s}^i + z_s^i)$ or $q_s\omega_{1s}^i + q_s z_s^i = 1/\lambda_i$. Summing over i and using asset market clearing, we find $4q_s = 1/\lambda_1 + 1/\lambda_2$. The q_s must be equal, so we may set $q_s = 1$ for every security s.

Now $1/\lambda_i = \omega_{1s}^i + z_s^i = \omega_{1s}^i + q_s z_s^i$. Summing over s and using the asset budget constraint we find $3/\lambda_i = 6$. Thus $1/\lambda_i = 2$, so $2 = \omega_{1s}^i + z_s^i$. It follows that $\mathbf{z}^1 = (1,0,-1)$ and $\mathbf{z}^2 = (-1,0,1)$. The corresponding goods allocations are $\mathbf{x}^1 = (2,2,2)$ and $\mathbf{x}^2 = (2,2,2)$.