1. Suppose a firm uses two inputs $z_1$ and $z_2$. The production function is $f(z) = 2z_1 + 3z_2$. The net output vector is $y = (-z_1, -z_2, q)$ where $q \leq f(z)$ and $z \geq 0$. The price vector is $(w_1, w_2, p)$.

   a) For what values of $(w_1, w_2, p)$ is it possible to maximize profit?

   **Answer:** For this CRS technology, profit can only be maximized if prices are in the dual cone of the production set. Free disposal then implies $(w_1, w_2, p) \geq 0$ when profit can be maximized. Maximum profit will occur when $q = f(z)$. Then profit is $pq - w_1 z_1 - w_2 z_2 = (2p - w_1) z_1 + (3p - w_2) z_2$. Since $z \geq 0$, profit maximization is only possible if $2p - w_1 \leq 0$ and $3p - w_2 \leq 0$. Thus profit can be maximized if and only if $p \leq \min\{w_1/2, w_2/3\}$.

   b) For what values of $(w_1, w_2, p)$ will the supply correspondence contain only 0?

   **Answer:** Using the expression for profit, we find that $z = 0$ is the only profit-maximizing input if and only if $p < \min\{w_1/2, w_2/3\}$. Then $q \leq 0$, so $p > 0$ is also required to insure $q = 0$. The condition is $0 < p < \min\{w_1/2, w_2/3\}$.

   c) For what values of $(w_1, w_2, p)$ will the supply correspondence contain vectors other than 0?

   **Answer:** If $p = 0$ and $0 \leq \min\{w_1/2, w_2/3\}$, vectors of the form $(0, 0, q)$ for $q < 0$ will be in the supply correspondence. If $p > 0$ and $p = w_1/2$, vectors with $q = z_1 > 0$ will be in the supply correspondence, while if $p = w_2/3$, vectors with $q = z_2 > 0$ will be in the supply correspondence.

2. Consider a two-person exchange economy with endowments $\omega^1 = (2, 1)$ and $\omega^2 = (2, 2)$. The utility functions are linear: $u_1(x^1) = x_1^1 + 3x_2^1$ and $u_2(x^2) = 2x_1^2 + x_2^2$. Find all Walrasian equilibria.

   **Answer:** There is more than one way to solve this problem. There is the brute force method of calculating the demand correspondences and using market clearing to find the equilibrium.

   A second method is to reason our way to the answer. The aggregate endowment is $(4, 3)$. Both consumers have linear indifference curves. Consumer 1’s indifference curves have slope $-2$ while consumer 2’s have slope $-1/3$. The shaded area shows the potential mutual improvements, which lies above and to the left of the endowment point. This means that the relative price of good 2 must be between 1/3 and 2 (the absolute slopes of the indifference curves through $\omega^1$).

   If the relative price $p = p_1/p_2$ is greater than 1/3, consumer one will be at a corner solution and consume only good 2 on the left boundary. If the relative price is less that 2, consumer two will be at a corner consuming only good 1 on the upper boundary. Note that the budget line with $p = 1/3$ (same as $u_1$) does not intersect the upper boundary inside the Edgeworth box and the budget line with $p = 2$ ($u_2$) does not intersect the side boundary in the Edgeworth box. This rules
out equilibria with \( p = 1/3 \) or \( p = 2 \). The only point that is both on the upper boundary and the left boundary is the corner point \( E = (0,3) \), and that is the equilibrium allocation.

The budget line must pass through both \( \omega^1 \) and \( E \), so the relative price is \( p_1/p_2 = (3-1)/(4-2) = 1 \). This gives equilibrium price vector \( p = (1,1) \) (and any scalar multiple). Consumer 1 has income 3 while consumer 2 has income 4. Consumer one can just afford \((3,0)\) and consumer two can just afford \((0,4)\), which is the equilibrium allocation.

3. Suppose \( e(p, \bar{u}) = p_1 + 2p_2 + 4\bar{u}\sqrt{p_1 p_2} \) is a consumer’s expenditure function.

   a) Find the indirect utility function \( v(p, m) \).

   **Answer:** By duality, \( e(p, v(p, m)) = m \). Thus \( p_1 + 2p_2 + 4v(p, m)\sqrt{p_1 p_2} = m \). It follows that \( v(p, \text{inc}) = (m - p_1 - 2p_2)/4\sqrt{p_1 p_2} \).

   b) Find the utility function \( u(x) \).

   **Answer:** We know that the conjugate of \( e \) will be the indicator of \( \{x : u(x) \geq \bar{u}\} \). Now consider \( p \cdot x - e(p, \bar{u}) = p_1(x_1 - 1) + p_2(x_2 - 2) - 4\bar{u}\sqrt{p_1 p_2} \). If \( x_1 < 1 \) or \( x_2 < 2 \) the infimum is \(-\infty\), so suppose \( x_1 \geq 1 \) and \( x_2 \geq 2 \). Now consider \( p \cdot x - e(p, \bar{u}) = (\sqrt{p_1(x_1 - 1)} - \sqrt{p_2(x_2 - 2)})^2 + 2\sqrt{p_1 p_2(x_1 - 1)(x_2 - 2)} - 4\bar{u}\sqrt{p_1 p_2} \). This will have minimum \(-\infty\) if \( \sqrt{(x_1 - 1)(x_2 - 2)} < 2\bar{u} \) and minimum 0 if \( \sqrt{(x_1 - 1)(x_2 - 2)} \geq 2\bar{u} \). It follows that utility has the Stone-Geary form \( u(x) = \frac{1}{2}\sqrt{(x_1 - 1)(x_2 - 2)} \) for \( x_1 \geq 1 \) and \( x_2 \geq 2 \).

   Another method is to compute the Hicksian demands \( h_1 = 1 + 2\bar{u}\sqrt{p_2/p_1} \) and \( h_2 = 2 + 2\bar{u}\sqrt{p_2/p_1} \). Then \( (h_1 - 1)/2\bar{u} = \sqrt{p_2/p_1} \) and \( (h_2 - 2)/2\bar{u} = \sqrt{p_1/p_2} \). Multiplying, we obtain \((h_1 - 1)(h_2 - 2)/4\bar{u}^2 = 1 \) which again yields the Stone-Geary utility \( u(x) = \frac{1}{2}\sqrt{(x_1 - 1)(x_2 - 2)} \).
4. Suppose utility is \( u(x_1, x_2, x_3) = x_1^2 + 2x_2x_3 + 2x_1x_2 + x_3^2 \).

\( a) \) For what partitions of \( \{1, 2, 3\} \) is \( u \) weakly separable?

**Answer:** Since \( u \) is increasing, it is weakly separable with respect to \( \mathcal{P} = \{\{1\}, \{2\}, \{3\}\} \). For other partitions, we compute the marginal rates of substitution: 
\[
\text{MRS}_{12} = \frac{x_1 + x_2}{x_1 + x_3}, \\
\text{MRS}_{13} = \frac{x_1 + x_2}{x_2 + x_3}, \\
\text{MRS}_{23} = \frac{x_1 + x_3}{x_2 + x_3}.
\]
In each case, the MRS depends on all three variables. This shows that \( u \) is not weakly separable on any partition other than \( \mathcal{P} = \{\{1\}, \{2\}, \{3\}\} \).

\( b) \) For what partitions of \( \{1, 2, 3\} \) is \( u \) strongly separable?

**Answer:** By (b), \( u \) is only separable on \( \mathcal{P} = \{\{1\}, \{2\}, \{3\}\} \). That means it is not strongly separable on any partition.

\( c) \) Is \( u \) completely separable?

**Answer:** No, since it is not strongly separable on any partition other than \( \cap \mathcal{P} = \{\{1\}, \{2\}, \{3\}\} \).