Homework #1

1.3.5 Compute the marginal rate of substitution MRS_{12} for the following utility functions.

a) The utility $u(\mathbf{x}) = \alpha_1 x_1 + \alpha_2 x_2$ where $\alpha_\ell > 0$.

b) The utility $v(\mathbf{x}) = \alpha_1 x_1 + \alpha_2 x_2^2$ where $\alpha_\ell > 0$.

Are the above utility functions equivalent? Explain.

Answer: For (a), the marginal rate of substitution is $\text{MRS}_{12} = \alpha_1/\alpha_2$ while for (b), the marginal rate of substitution is $\text{MRS}_{12} = \alpha_1/(2\alpha_2 x_2)$. Since these are not the same, the utility functions cannot be equivalent.

2.2.2 Let $u\colon \mathbb{R}_+ \to \mathbb{R}$ be defined by $u(x) = x$ for $x < 1$ and $u(x) = x + 1$ for $x \geq 1$. Show that the preference order defined by u is a continuous preference order.

Answer: We must show that both the upper and lower contour sets are closed. Now

$$\{x : u(x) \geq u^0\} = \begin{cases} [u^0 - 1, +\infty) & \text{if } u^0 \geq 2 \\ [1, +\infty) & \text{if } 1 \leq u^0 \leq 2 \\ [u^0, +\infty) & \text{if } 0 \leq u^0 \leq 1 \end{cases}$$

so the upper contour set is closed. Similarly,

$$\{x : u(x) \leq u^1\} = \begin{cases} [0, u^1 - 1] & \text{if } u^1 \geq 2 \\ [0, 1] & \text{if } 1 \leq u^1 \leq 2 \\ [0, u^1] & \text{if } 0 \leq u^1 \leq 1 \end{cases}$$

so the lower contour set is closed. It follows that the preference order is continuous.

2.3.1 Consider the Cobb-Douglas utility $u(\mathbf{x}) = x_1^\alpha x_2^\beta$ where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Use the Hessian to show that u is a strictly concave function for $x_1, x_2 > 0$.

Answer: We compute the Hessian matrix

$$\mathbf{H} = \begin{pmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^\beta & \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} & \beta(\beta-1)x_1^\alpha x_2^{\beta-2} \end{pmatrix}.$$

Now $0 < \alpha < 1$, so the first leading principal minor $H_1 = \alpha(\alpha-1)x_1^{\alpha-2}x_2^\beta < 0$ and the second leading principal minor $H_2 = [\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2]x_1^{2\alpha-2}x_2^{2\beta-2} = \alpha\beta(1-\alpha-\beta)x_1^{2\alpha-2}x_2^{2\beta-2} > 0$. This shows that H is negative definite, implying that u is strictly concave.

2.5.4 Let $u \in \mathcal{C}^2$ be a utility function on $\mathbb{R}^2_+$ with $\partial u/\partial x_1, \partial u/\partial x_2 > 0$. Show that u is completely separable. This implies that Corollary 2.5.13 fails when $L = 2$.

Answer: Since u is increasing in each argument, it induces an order on $\{1\}$ and $\{2\}$. Since the only possible partitions of $\{1,2\}$ are $\{\{1\},\{2\}\}$ and $\{1,2\}$, it is strongly separable on $\{1,2\}$ relative to the partition of singletons. This means it is completely separable. However, as shown in Exercise 2.5.3, such functions need not be additively separable.

2.5.5 Let $u(\mathbf{x}) = x_1^2 + 2x_1x_2x_3 + x_2^2x_3^2$. Is u separable on $\mathbb{R}^3_{++}$ relative to any partition? If so, is u strongly separable relative to that partition? Does u have a quasi-linear representation?

Answer: Since u is increasing, it is separable relative to the partition $\big\{\{1\},\{2\},\{3\}\big\}$. For the rest, it will be helpful to calculate the marginal rates of substitution. They are $\text{MRS}_{12} = 1/x_3$, $\text{MRS}_{13} = 1/x_2$ and $\text{MRS}_{23} = x_3/x_2$. Since MRS_{12} depends on x_3, which is not in $\{1\} \cup \{2\}$, u is not strongly separable relative to $\big\{\{1\},\{2\},\{3\}\big\}$.

The same marginal rate of substitution also tells us that u is not separable relative to $\big\{\{1,2\},\{3\}\big\}$. It is also not separable relative to $\big\{\{1,3\},\{2\}\big\}$ because $\text{MRS}_{13} = 1/x_2$. The lack of separability in the last two cases implies there are no quasi-linear representations relative to x_2 or x_3

That leaves $\big\{\{1\},\{2,3\}\big\}$, which passes the marginal rate of substitution test. In fact, $u(\mathbf{x}) = (x_1 + x_2x_3)^2$, and is equivalent to $v(\mathbf{x}) = \sqrt{u(\mathbf{x})} = x_1 + x_2x_3$. Regardless of the value of x_1, the ranking of (x_1, x_2, x_3) and (x_1, y_2, y_3) only depends on whether x_2x_3 is bigger than y_2y_3. These preferences are both separable and strongly separable relative to $\big\{\{1\},\{2,3\}\big\}$. Moreover, we have quasi-linear representation relative to x_1, $v(\mathbf{x}) = x_1 + x_2x_3$.