Homework #2

4.1.5 Suppose utility is defined on $\mathbb{R}^L_+$ by the Leontief utility $u(\mathbf{x}) = \min\{\alpha_i x_i : i = 1, \dots, L\}$ with each $\alpha_i > 0$. Find the expenditure function and compensated demands for $\mathbf{p} \gg \mathbf{0}$.

Answer: We minimize over the set $\{\mathbf{x} \in \mathbb{R}^L_+ : u(\mathbf{x}) \geq \bar{u}\}$. This requires $\alpha_i x_i \geq \bar{u}$ for each i. Expenditure will reach its minimum when every constraint binds, so $x_i = \bar{u}/\alpha_i$. Thus the Hicksian demand is always a function and is given by $h_i(\mathbf{p}, \bar{u}) = \bar{u}/\alpha_i$. It follows that the expenditure function is $e(\mathbf{p}, \bar{u}) = \sum_i p_i h_i(\mathbf{p}, \bar{u}) = \bar{u} \sum_i p_i/\alpha_i$.

4.2.1 Suppose $e(\mathbf{p}, \bar{u}) = (p_1 + 2p_2)\sqrt{\bar{u}}$. Find the Hicksian demands. Then find the underlying utility function.

Answer: The Shephard-McKenzie Lemma tells us that $\mathbf{h}(\mathbf{p}, \bar{u}) = \sqrt{\bar{u}}(1, 2)$. Here $h_1 = \sqrt{\bar{u}}$ and $h_2 = 2h_1 = 2\sqrt{\bar{u}}$. Thus $\bar{u} = h_1^2 = h_2^2/4$. The utility function is $u(\mathbf{x}) = \min\{x_1^2, x_2^2/4\}$.

4.2.3 Suppose indirect utility has the Gorman form $v(\mathbf{p}, m) = a(\mathbf{p}) + mb(\mathbf{p})$ for some functions $a(\mathbf{p})$ and $b(\mathbf{p})$.

a) Show that the expenditure function has a similar form, $e(\mathbf{p}, \bar{u}) = c(\mathbf{p}) + \bar{u}d(\mathbf{p})$ for some functions $c(\mathbf{p})$ and $d(\mathbf{p})$.

b) What degrees of homogeneity must c and d have for Theorem 4.1.5 to hold?

c) What does (b) imply about the homogeneity of a and b? Is this consistent with Theorem 3.4.1?

Answer:

a) Using duality, we may write $\bar{u} = v(\mathbf{p}, e(\mathbf{p}, \bar{u}) = a(\mathbf{p}) + mb(\mathbf{p})$. Solving for $e(\mathbf{p}, \bar{u})$, we find $e(\mathbf{p}, \bar{u}) = c(\mathbf{p}) + \bar{u}d(\mathbf{p})$ where $c(\mathbf{p}) = -a(\mathbf{p})/b(\mathbf{p})$ and $d(\mathbf{p}) = 1/b(\mathbf{p})$.

b) For e to be homogeneous of degree 1 in $\mathbf{p}$, both c and d must be homogeneous of degree 1 in $\mathbf{p}$ (set $\bar{u} = 0$ to see this for c, it then follows for d).

c) Then b must be homogeneous of degree -1 in $\mathbf{p}$ while a is homogeneous of degree 0. This implies v is homogeneous of degree 0 in $(\mathbf{p}, m)$, as required by the theorem.

5.1.1 Let $A = \{(x, y) : y \geq x^2\}$ and $\mathbf{b} = (2, 1/2)$.

a) Show that A is a convex set and that $\mathbf{b} \notin A$.

b) Find the point in A that is closest to $\mathbf{b}$. That is, minimize $\|\mathbf{x} - \mathbf{b}\|^2$ where $\mathbf{x} = (x, y)$ obeys $y \geq x^2$. Then find a $\mathbf{p}$ that separates $\mathbf{b}$ from A as in Separation Theorem A.

Answer:

a) Here $A = \{(x,y) : 0 \leq y - x^2\}$. Since $y - x^2$ is concave, A is convex as an upper contour set of a concave function. Since $1/2 < 2^2$, $\mathbf{b} = (2, 1/2) \notin A$.

b) We must minimize $(x-2)^2 + (y-1/2)^2$ subject to the constraint $y - x^2 \leq 0$. The Lagrangian is $\mathcal{L} = (x-2)^2 + (y-1/2)^2 - \lambda(y - x^2)$. The first order conditions are $2(x-2) = -2\lambda x$ and $2(y-1/2) - \lambda = 0$. Clearly, $x \neq 0$, so we can write $\lambda = -(x-2)/x$. Further, $\lambda \neq 0$, so $y = x^2$. Putting this together, we obtain $2(x^2 - 1/2) + (x-2)/x = 0$. This can be rewritten as $2x^3 - x + x - 2 = 0$, so $x^3 = 1$. The solution is $\mathbf{w} = (1,1)$. Then $\mathbf{p} = \mathbf{b} - \mathbf{w} = (1, -1/2)$ separates A and $\mathbf{b}$.

5.1.3 Let $A = \{(x,y) \in \mathbb{R}^2_+ : x^{1/3}y^{2/3} \geq 3\}$ and $\mathbf{x}_0 = (27, 1)$.

a) Show that A is a convex set.

b) Show that $\mathbf{x}_0 \notin \operatorname{int} A$.

c) Use Support Property II from Chapter 22 to find a vector $\mathbf{p}$ as in Separation Theorem C.

Answer:

a) The set A is an upper contour set of a concave function, so it is convex.

b) Now $(27)^{1/3}(1)^{2/3} = 3$, so $\mathbf{x}_0 = (27,1)$ is on the boundary of A. Note that $\operatorname{int} A = \{(x,y) \in \mathbb{R}^2_+ : x^{1/3}y^{2/3} > 3\}$.

c) Let $f(x,y) = x^{1/3}y^{2/3}$. Support Property II tells us that $df(\mathbf{x}_0)\cdot\mathbf{x} \geq df(\mathbf{x}_0)\cdot\mathbf{x}_0$ for all $\mathbf{x} \in A$. Now $df = (x^{-2/3}y^{2/3}/3, 2x^{1/3}y^{-1/3}/3)$. Thus $df(\mathbf{x}_0) = (1/27, 2)$, so $(1/27, 2)\cdot\mathbf{x} \geq 3$ for all $\mathbf{x} \in A$.