Homework #3

5.2.1 Let $g(\mathbf{x}) = f(\mathbf{x} - \mathbf{a})$ for some $\mathbf{a} \in \mathbb{R}^L$. Show that $g^*(\mathbf{p}) = \mathbf{p}\cdot\mathbf{a} + f^*(\mathbf{p})$.

Answer: Here

$$\begin{aligned} g^*(\mathbf{p}) &= \inf\{\mathbf{p}\cdot\mathbf{x} - g(\mathbf{x})\} \\ &= \inf_{\mathbf{x}}\{\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x} - \mathbf{a})\} \\ &= \inf_{\mathbf{x}}\{\mathbf{p}\cdot(\mathbf{x} + \mathbf{a}) - f(\mathbf{x})\} \\ &= \mathbf{p}\cdot\mathbf{a} + \inf_{\mathbf{x}}\{\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x})\} \\ &= \mathbf{p}\cdot\mathbf{a} + f^*(\mathbf{p}). \end{aligned}$$

5.4.3 Let $f(x) = -e^{-x}$. Compute f^*.

Answer: We use the Legendre transformation. We solve $df/dx = e^{-x} = p$ for $x(p) = -\ln p$ whenever $p > 0$. Now $f^*(p) = px(p) - f\big(x(p)\big) = -p\ln p + p$. The definition of f^* tells us $f^*(0) = 0$ and $f^*(p) = -\infty$ for $p < 0$. In sum,

$$f^*(p) = \begin{cases} -p\ln p + p & \text{when } p > 0 \\ 0 & \text{when } p = 0 \\ -\infty & \text{when } p < 0. \end{cases}$$

5.5.2 Let $e(\mathbf{p}, \bar{u}) = \bar{u}(p_1 + 3p_2)$. Find the conjugate function $e^*_{\bar{u}}(\mathbf{x})$.

Answer: We must minimize $\mathbf{p}\cdot\mathbf{x} - e(\mathbf{p}, \bar{u}) = p_1(x_1 - \bar{u}) + p_2(x_2 - 3\bar{u})$. Thus

$$e^*_{\bar{u}}(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1 \ge \bar{u} \text{ and } x_2 \ge 3\bar{u} \\ -\infty & \text{otherwise.} \end{cases}$$

Although not part of the problem, it is easy to see that the associated quasiconcave utility function is $u(\mathbf{x}) = \min\{x_1, x_2/3\}$.

5.6.2 Let utility be $u(x_1, x_2) = 2x_1 - 3x_2$ on the consumption set $\mathbb{R}^2_+$.

a) Find $e(\mathbf{p}, \bar{u})$ for all $\mathbf{p} \in \mathbb{R}^2$ (not just $\mathbf{p} \in \mathbb{R}^2_+$).

b) For each $\mathbf{p} \gg \mathbf{0}$ solve $m = e\big(\mathbf{p}, v(\mathbf{p}, m)\big)$ to find $v(\mathbf{p}, m)$ and then compute the Marshallian demand $\mathbf{x}(\mathbf{p}, m)$.

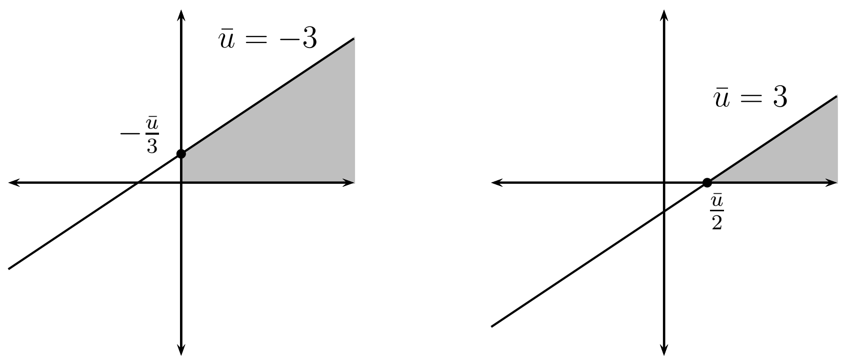
c) Let u^M be the monotone hull of u and $e^M(\mathbf{p}, \bar{u})$ be the corresponding expenditure function. Compute both u^M and $e^M(\mathbf{p}, \bar{u})$.

d) For each $\mathbf{p} \gg \mathbf{0}$ solve $m = e^M\big(\mathbf{p}, v(\mathbf{p}, m)\big)$ to find $v(\mathbf{p}, m)$ and then compute the Marshallian demand $\mathbf{x}^M(\mathbf{p}, m)$.

e) Are the Marshallian demands $\mathbf{x}(\mathbf{p}, m)$ and $\mathbf{x}^M(\mathbf{p}, m)$ the same for $\mathbf{p} \gg \mathbf{0}$? Is $e(\mathbf{p}, \bar{u}) = e^M(\mathbf{p}, \bar{u})$ when $\mathbf{p} \gg \mathbf{0}$?

Answer:

It helps to draw a diagram or two to understand the set we are minimizing over when we find the expenditure function.



The left-hand panel is typical of the cases where $\bar{u} < 0$ and the right-hand panel exemplifies the cases wheren $\bar{u} > 0$.

a) By Proposition 5.6.1, $e(\mathbf{p}, \bar{u}) = -\infty$ when $p_1 < 0$. Even without the proposition, the diagrams make this clear since $(x_1, 0)$ is feasible for arbitrarily large x_1.

This leaves the cases where $p_1 \geq 0$. We form the Lagrangian $\mathcal{L} = p_1x_1 + p_2x_2 - \lambda(2x_1 - 3x_2 - \bar{u}) - \mu_1x_1 - \mu_2x_2$. The resulting first-order conditions are

$$p_1 = 2\lambda + \mu_1$$
$$p_2 = -3\lambda + \mu_2.$$

The constraint is that $2x_1 - 3x_2 \geq \bar{u}$, or $2x_1 \geq \bar{u} + 3x_2$.

As the diagram suggests, we should separately consider the cases $\bar{u} < 0$ and $\bar{u} > 0$.

We start with the $\bar{u} > 0$ case. Since $x_2 \geq 0$, $x_1 > 0$ when $\bar{u} > 0$. This implies $\mu_1 = 0$ by complementary slackness, so $\lambda = p_1/2$ and $p_2 = -3p_1 + \mu_2$. If $p_2 > -3p_1$, we have $\mu_2 > 0$, implying that $x_2 = 0$. But then $2x_1 \geq \bar{u}$ and $p_1x_1 \geq p_1\bar{u}/2$. We also obtain $e(\mathbf{p}, \bar{u}) = p_1\bar{u}/2$ if $\mu_2 = 0$. This is obvious on the diagram.

Finally, if $p_2 < -3p_1$, $\mu_2 < 0$ which is impossible. In this case there is no minimum and $e(\mathbf{p}, \bar{u}) = -\infty$. Another way to see this is to use the diagram. There $(1, -3)$ is perpendicular to the utility constraint. If p_2 is more negative lie below the ray defined by $(1,-3)$. In that case the isoexpenditure lines cut the utility constraint and higher lines have more negative expenditure.

If $\bar{u} \le 0$, we can no longer conclude $\mu_1 = 0$. However, it is now the case that $(0,0)$ is feasible, so if $\mathbf{p} \ge \mathbf{0}$, $e(\mathbf{p},\bar{u}) = 0$. That leaves the case where $p_1 \ge 0$ and $p_2 < 0$, where the price vector is in the SE quadrant.

The diagram again tells us that if price vector is above the ray defined by $(1,-3)$, the minimum is at the upper left corner, $(0,-\bar{u}/3)$. If the price vector is below that ray, there is no minimum and $e(\mathbf{p},\bar{u}) = -\infty$.

Alternatively, using the first-order conditions, we have $\lambda > 0$ so $2x_1 - 3x_2 = \bar{u}$. Now if $\bar{u} < 0$, $x_2 < 0$, implying $\mu_2 = 0$ by complementary slackness. Thus $p_1 = -2p_2/3 + \mu_1$. If $p_1 > -2p_2/3$, $\mu_1 > 0$ implying $x_1 = 0$ by complementary slackness. But then $x_2 = -\bar{u}/3$ so the minimum is $-p_2\bar{u}/3$. If $p_1 < -2p_2/3$, there is no solution $(e(\mathbf{p},\bar{u}) = -\infty)$ and if $p_1 = -2p_2/3$, $p_1x_1 + p_2x_2 = -p_2\bar{u}/3$.

Summing up, for $\bar{u} > 0$, we have

$$e(\mathbf{p},\bar{u}) = \begin{cases} p_1\bar{u}/2 & \text{when } p_1 \ge 0 \text{ and } p_2 \ge -3p_1/2 \\ -\infty & \text{otherwise.} \end{cases}$$

and for $\bar{u} \le 0$, the expenditure function is

$$e(\mathbf{p},\bar{u}) = \begin{cases} -p_2\bar{u}/3 & \text{when } p_1 \ge 0 \text{ and } p_2 \ge -3p_1/2 \\ -\infty & \text{otherwise.} \end{cases}$$

b) When $\mathbf{p} \gg \mathbf{0}$, $v(\mathbf{p},m) \ge 0$. It follows that $m = e\big(\mathbf{p}, v(\mathbf{p},m)\big) = p_1 v(\mathbf{p},m)/2$. Then $v(\mathbf{p},m) = 2m/p_1$. Roy's identity yields $\mathbf{x}(\mathbf{p},m) = (m/p_1, 0)$.

c) Now $u^M(\mathbf{x}) = \sup\{2y_1 - 3y_2 : \mathbf{0} \le \mathbf{y} \le \mathbf{x}\} = 2x_1$. Then $e(\mathbf{p},\bar{u}) = p_1\bar{u}/2$ for $\mathbf{p} \ge \mathbf{0}$. Of course $e(\mathbf{p},\bar{u}) = -\infty$ for $\mathbf{p} \not\ge \mathbf{0}$.

d) Here $m = p_1 v(\mathbf{p},m)/2$, so $v^M(\mathbf{p},m) = 2m/p_1$. Roy's identity then yields $\mathbf{x}(\mathbf{p},m) = (m/p_1, 0)$.

e) Yes, both are the same.