Homework #4

7.1.2 Suppose consumers have indirect utility of the form $v_i(\mathbf{p}, m_i) = a_i(\mathbf{p})^2 + 2a_i(\mathbf{p})b(\mathbf{p})m + m^2b(\mathbf{p})^2$ for $m_i \in M_i$, an open interval, with the function a homogeneous of degree zero in $\mathbf{p}$ and the function b homogeneous of degree -1 in $\mathbf{p}$. Is strong aggregation possible here?

Answer: **Yes**. Note that $v_i(\mathbf{p}, m_i) = \big(a_i(\mathbf{p}) + mb(\mathbf{p})\big)^2$. This is equivalent to the indirect utility function $w_i(\mathbf{p}, m_i) = a_i(\mathbf{p}) + mb(\mathbf{p})$, which allows strong aggregation by Proposition 7.1.1.

A second way to obtain the result is to use Roy's Identity. Here

$$\frac{\partial v_i}{\partial p_\ell} = 2(a_i + mb)\left(\frac{\partial a_i}{\partial p_\ell} + m\frac{\partial b}{\partial p_\ell}\right)$$

and

$$\frac{\partial v_i}{\partial m} = 2(a_i + mb)\, b(\mathbf{p}).$$

Roy's Identity yields

$$x_\ell = -\frac{1}{b}\frac{\partial a_i}{\partial p_\ell} - m\frac{1}{b}\frac{\partial b}{\partial p_\ell},$$

which has the necessary form.

8.2.4 Suppose the production function can be written $f(\mathbf{z}) = F\big(\sum_\ell \phi_\ell(z_\ell)\big)$. Show that the marginal rate of technical substitution between any two inputs is independent of the amount of any other input used.

Answer: The marginal rate of technical substitution between k and ℓ is

$$\text{MRTS}_{k\ell} = \frac{\partial F/\partial z_k}{\partial F/\partial z_\ell} = \frac{F'\phi_k'(z_k)}{F'\phi_l'(z_\ell)} = \frac{\phi_k'(z_k)}{\phi_\ell'(z_\ell)},$$

which depends only on z_k and z_ℓ.

8.2.5 Consider the cost minimization problem for $f(z_1, z_2) = \sqrt{z_1 z_2}$ and $\mathbf{w} = (1, 0)$. Show that the cost minimization problem does not have a solution for any $q > 0$.

Answer: For $n > 0$, the input $\mathbf{z} = (q/n, qn)$ yields output q and its cost is $q/n \to 0$ as $n \to \infty$. The infimum of cost is zero, but this is never realized for any $\mathbf{z}$ with $\sqrt{z_1 z_2} \geq q$.

9.1.2 Find sets obeying the following conditions:

a) The set fails the no free lunch condition.

b) The set is a production set that is additive, but not divisible.

c) The set is a production set that is not convex.

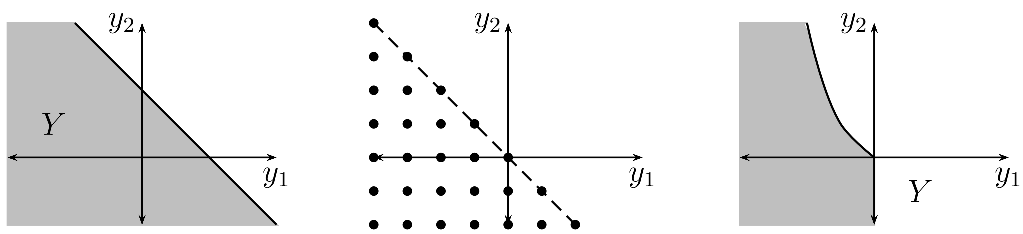
Answer: Many such examples are possible.

a) The set $Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1 + y_2 \le 2\}$ is a set that fails the no free lunch condition (P3) because $(1,1) \in Y$. Notice that it obeys the other conditions for a production set: It is non-empty, closed, and obeys inaction and free disposal.

b) The set $Y = \{\mathbf{y} \in \mathbb{R}^2 : y_1, y_2$ are non-positive integers with $y_1 + y_2 \le 0\}$ is such a set. The set Y is additive because the sum of non-positive integers is non-positive. Since Y consists of isolated points, it is not divisible. E.g., $\frac{1}{2}(-1,-1) = (-\frac{1}{2}, -\frac{1}{2}) \notin Y$. It is easy to see it obeys (P1)–(P5), and so is a production set.

c) Consider the production function $f(z) = z^2$. This increasing returns to scale production function yields a production set that fails convexity. Let $Y = \{(-z, q) : q \le z^2, z \le 0\}$. Then $(-1,1) \in Y$, but $\frac{1}{2}(-1,1) + \frac{1}{2}(0,0) = (-\frac{1}{2}, \frac{1}{2}) \notin Y$ because $\frac{1}{2} \not\le (\frac{1}{2})^2 = \frac{1}{4}$. Since f is continuous and $f(0) = 0$, we showed in Example 9.1.1 that Y is a production set.

These are illustrated in the figures below. Note that examples used in (b) and (c) apply to both cases.



(a) **(b)** **(c)**