Homework #5

10.2.1 Suppose production is described by a linear activity model with three activities: $\mathbf{a}^1 = (3, 0, -1)^\mathbf{T}$, $\mathbf{a}^2 = (0, 2, -1)^\mathbf{T}$ and $\mathbf{a}^3 = (1, 1, -1/2)^\mathbf{T}$.

a) Find the production set's dual cone and find a non-trivial price vector in the dual cone.

b) At what prices in the dual cone may activity 1 be utilized?

c) At what prices in the dual cone may activity 2 be utilized?

d) At what prices in the dual cone may activity 3 be utilized?

e) At what prices in the dual cone may both activities 1 and 2 be utilized?

f) Are there any non-zero prices in the dual cone where all activities may be utilized?

Answer:

a) We need to find price vectors with $\mathbf{p}\cdot\mathbf{a}^i \le 0$ for $i = 1, 2, 3$. This requires

$$\begin{aligned} 3p_1 - p_3 &\le 0 \\ 2p_2 - p_3 &\le 0 \\ p_1 + p_2 - p_3/2 &\le 0. \end{aligned}$$

A non-trivial price vector in the dual cone is $\mathbf{p} = (1/4, 1/3, 2)^\mathbf{T}$. Note that $\mathbf{p}\cdot\mathbf{a}^i \le 0$ for all i.

b) For the next three parts, we need to satisfy the zero profit condition for the activity in question. In the case of activity 1, this requires $3p_1 = p_3$.

c) This requires $2p_2 = p_3$.

d) This requires $p_1 + p_2 = p_3/2$.

e) Only price vectors proportional to $(2, 3, 6)^\mathbf{T}$ satisfy this. They are the only vectors in the dual cone that obey the conditions in both parts (b) and (c).

f) No. The vectors in part (e) fail to satisfy part (d).

11.3.1 Consider a two-person, two-good exchange economy. Endowments are $\boldsymbol{\omega}^1 = (1, 2)$ and $\boldsymbol{\omega}^2 = (1, 3)$ and utility is $u_1(\mathbf{x}^1) = x_1^1 + 2x_2^1$ and $u_2(\mathbf{x}^2) = \sqrt{x_1^2 x_2^2}$. Find all Walrasian equilibrium prices and allocations.

Answer: Since consumer two has Cobb-Douglas utility, both prices will have to be positive in equilibrium. We normalize so $(p_1, p_2) = (1, p)$. Then consumer incomes are

$m^1 = 1 + 2p$ and $m_2 = 1 + 3p$. Consumer two has equally weighted Cobb-Douglas utility, so $\mathbf{x}^2(p) = (1 + 3p)(1/2, 1/2p)$. Consumer one has linear utility, and will be at a corner unless $p = 2$. Consumer one's demand is:

$$\mathbf{x}^1(p) = \begin{cases} (1+2p, 0) & \text{if } p > 2 \\ \{(x, (5-x)/2) : 0 \le x \le 5\} & \text{if } p = 2 \\ (0, (1+2p)/p) & \text{if } p < 2. \end{cases}$$

where we have used the fact that $m_1 = 5$ when $p = 2$.

The aggregate endowment is $\boldsymbol{\omega} = (2, 5)$. If $p > 2$, only consumer two demands good 2 and market clearing requires $(1 + 3p)/2p = 5$. Then $p = 1/7$, contradicting the fact that $p > 2$. If $p < 2$, only consumer one demands good 1 and market clearing requires $(1+3p)/2 = 2$. Then $p = 1$ is an equilibrium. The corresponding allocation is $\mathbf{x}^1 = (0, 3)$ and $\mathbf{x}^2 = (2, 2)$. Finally, if $p = 2$, $\mathbf{x}^2 = (7/2, 7/4)$, meaning there is excess demand for good 1. This is not an equilibrium.

11.4.1 Consider a two-agent, two-good, one-firm production economy where utility is $u_1(\mathbf{x}^1) = (x_1^1)^{1/2}(x_2^1)^{1/2}$ and $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$, and endowments are $\boldsymbol{\omega}^1 = (3, 0)$ and $\boldsymbol{\omega}^2 = (6, 0)$. There is one firm with production set $Y = \{\mathbf{y} : y_1 \le 0, y_2 \le -y_1\}$. Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

Answer: If the firm does not produce anything, the economy will not have any of good 2. Since demand for good 2 will be positive, this cannot be an equilibrium.

If the firm does produce ($y_2 > 0$), constant returns to scale implies that profits will be zero. This requires $p_1 = p_2$. We can now normalize prices so $\mathbf{p} = (1, 1)$. Incomes are $m^1 = 3$ and $m^2 = 6$. The Cobb-Douglas utilities yield demands $\mathbf{x}^1 = (3/2, 3/2)$ and $\mathbf{x}^2 = (2, 4)$. Now $\mathbf{x}^1 + \mathbf{x}^2 = \boldsymbol{\omega} + \mathbf{y}$ by market clearing. This can be rewritten $(7/2, 11/2) = (9, 0) + (-y_2, y_2)$. It follows that the production vector is $(-11/2, +11/2)$.

11.4.3 Consider a two-agent, two-good, one-firm production economy where utility is $u_1(\mathbf{x}^1) = (x_1^1)^{1/2}(x_2^1)^{1/2}$ and $u_2(\mathbf{x}^2) = (x_1^2)^{1/2}(x_2^2)^{1/2}$, and endowments are $\boldsymbol{\omega}^1 = (4, 0)$ and $\boldsymbol{\omega}^2 = (4, 0)$. Each agent will receive half of the profits of the firm. There is one firm with production set $Y = \{\mathbf{y} : y_1 \le 0, y_2 \le \sqrt{-y_1}\}$. Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

Answer: We start by considering profit, $p_2\sqrt{-y_1} + p_1 y_1$. The first-order conditions for profit maximization are $(p_2/2)(1/\sqrt{-y_1}) = p_1$, so $-p_2^2/4p_1^2 = y_1$, $p_2/2p_1 = y_2$, and $\pi(\mathbf{p}) = p_2^2/4p_1$. Each consumer receives half of the profit and has endowment income

$4p_1$, so each consumer has income $m = 4p_1 + p_2^2/8p_1$. Since consumers have indentical preferences, and income, they will consume the same amount. Utility is equal-weighted Cobb-Douglas, and market demand is $2m(1/2p_1, 1/2p_2) = m(1/p_1, 1/p_2)$. Adding the endowment to the firm's supply yields $(8 - p_2^2/4p_1^2, p_2/2p_1)$. Both goods are demanded in equilibrium, and prices must be strictly positive. We normalize prices so that $p_1 = 1$ and $p_2 = p$. Then market clearly for good 2 says $p/2 = m/p = (4+p^2/8)/p$. Thus $4p^2 = 32+p^2$ implying $p = \pm\sqrt{32/3}$. Only the positive root makes economic sense, and the equilibrium has $\mathbf{p} = (1, 4\sqrt{2/3})$, $\mathbf{y} = (-8/3, \sqrt{8/3})$, $m = 16/3$, and $\mathbf{x}^1 = \mathbf{x}^2 = (8/3, \sqrt{2/3})$.