Homework #6

14.1.1 Suppose $u_1(\mathbf{x}^1) = (x_1^1)^{1/3}(x_2^1)^{2/3}$, $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$, and $\boldsymbol{\omega} = (10, 2)$. Find all Pareto optimal allocations.

Answer: Since both consumers have strictly monotonic preferences, all goods will be consumed. Thus we may presume that $x_1^2 = 10 - x_1^1$ and $x_2^2 = 2 - x_2^1$. Further, because both consumers have Cobb-Douglas preferences, the only corner solutions will the be obvious ones ($\mathbf{x}^i = \boldsymbol{\omega}$ and $\mathbf{x}^j = \mathbf{0}$ for $i \neq j$). This means that we can use the simplified Lagrangian $\mathcal{L} = (x_1^1)^{1/3}(x_2^1)^{2/3} + \lambda\big((10 - x_1^1)^{1/3}(2 - x_2^1)^{2/3} - \bar{u}_2\big)$.

The first-order conditions are then

$$\begin{aligned} \frac{1}{3}\left(\frac{x_2^1}{x_1^1}\right)^{2/3} &= \lambda\frac{1}{3}\left(\frac{2 - x_2^1}{10 - x_1^1}\right)^{2/3} \\ \frac{2}{3}\left(\frac{x_1^1}{x_2^1}\right)^{1/3} &= \lambda\frac{2}{3}\left(\frac{10 - x_1^1}{2 - x_2^1}\right)^{1/3} \end{aligned}$$

Eliminating λ, we find

$$\frac{x_2^1}{x_1^1} = \frac{2 - x_2^1}{10 - x_1^1},$$

which implies $5x_2^1 = x_1^1$.

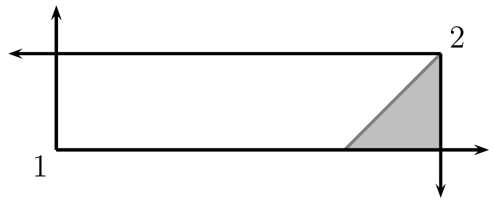
The Pareto set is

$$\left\{\alpha\begin{pmatrix}10\\2\end{pmatrix}, (1-\alpha)\begin{pmatrix}10\\2\end{pmatrix} : 0 \leq \alpha \leq 1\right\}.$$

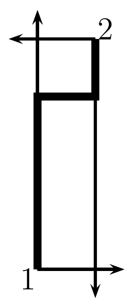
The corresponding utility levels are $u_1 = 2 \cdot 5^{1/3}\alpha$ and $u_2 = 2 \cdot 5^{1/3}(1 - \alpha)$.

14.1.4 Suppose $u_1(\mathbf{x}^1) = \max\{x_1^1, x_2^1\}$, $u_2(\mathbf{x}^2) = \min\{x_1^2, x_2^2\}$, and $\boldsymbol{\omega} = (4, 1)$. Find all Pareto optimal allocations. Be careful, u_1 is not a typo!

Answer: Since there is only one unit of good two, consumer two's utility can be at most one. But then there are at least 3 units of good one that cannot contribute to consumer two's utility. In that case, giving the excess to consumer one will be a Pareto improvement as one's utility from good two is bounded by 1. In other words, if consumer two receives x_1^2 at a Pareto optimum, consumer one must receive $x_1^1 = 4 - x_1^2$. Consumer two will also need $x_2^2 \geq x_1^2$, while consumer one doesn't care about good one. It follows that the Pareto set is $\{\big((x, y), (4-x, 1-y)\big) : 3 \leq x \leq 4, 1 \geq 1-y \geq 4-x\}$, as illustrated in the diagram below.



14.1.7 Suppose consumer one has the linear utility function $u_1(\mathbf{x}^1) = 2x_1^1 + x_2^1$, while consumer two has quasi-linear utility $u_2(\mathbf{x}^2) = x_1^2 + \sqrt{x_2^2}$. The endowment is $\boldsymbol{\omega} = (1, 4)$. Find all Pareto optimal allocations. **Answer:** Here $\mathrm{MRS}_{12}^1 = 2$ and $\mathrm{MRS}_{12}^2 = 2\sqrt{x_2^2}$. The interior solutions have $x_2^2 = 1$, so $x_2^1 = 3$. Good one can be allocated in any fashion. There are two kinds of boundary solutions. One where $x_2^2 > 1$. In that case, all of good one must be allocated to consumer two because of the larger MRS. Good two can be allocated in any fashion, as long as $x_2^2 > 1$. The other boundary case has $x_2^2 < 1$. Then MRS^1 is larger, so consumer 1 gets all of good one while good two can be allocated in any fashion, so long as $x_2^2 < 1$. The Pareto set is illustrated by the heavy lines in the diagram below.



15.1.2 An economy has two goods and two identical Cobb-Douglas consumers with $u_i(\mathbf{x}^i) = \sqrt{x_1^i x_2^i}$. The total endowment is $(0, 6)$. There is one constant returns to scale firm that produces good 1 and uses good 2 as its only input. The production function is $f(z) = 5z$.

a) Find all Pareto optimal allocations of goods and the corresponding net output vector.

b) Find prices and wealth levels that make each Pareto optima a quasi-equilibrium with taxes and transfers.

Answer:

a) The net output vector will have the form $\mathbf{y} = (-5y_2, y_2)$ with $y_2 \leq 0$. Here the marginal rate of transformation is $\mathrm{MRT}_{12} = 1/5$. This must also be the marginal rate of substitution at any interior Pareto optimum. Note that since utility is zero if there is no production, the production technology must be used.

Now $\mathrm{MRS}_{12}^i = x_2^i / x_1^i = 1/5$, so $5x_2^i = x_1^i$. Summing over both consumers, aggregate consumption obeys $5x_2 = x_1$. Since good 1 can only be obtained from the production sector, $x_1 = -5y_2$ and $x_2 = 6 + y_2$. Thus $x_2 = -y_2$ and $x_2 = 6 + y_2$. It follows that $x_2 = 3$ and $y_2 = -3$, so $x_1 = 15$.

Since both consumers will consume in the same proportions, $\mathbf{x}_1 = \alpha(15, 3)$ and $\mathbf{x}_2 = (1 - \alpha)(15, 3)$ for some α between 0 and 1. Also, $\mathbf{y} = (15, -3)$. These are the Pareto optimal allocations. The cases $\alpha = 0$ and $\alpha = 1$ are the Pareto optima on the border of the Edgeworth box.

b) We have already derived implicitly the prices in part (a), $\mathbf{p} = (1, 5)$. The wealth levels that yield the goods allocation $\mathbf{x}^1 = \alpha(15, 3)$ and $\mathbf{x}^2 = (1 - \alpha)(15, 3)$ are $m_1 = 30\alpha$ and $m_2 = 30(1 - \alpha)$.