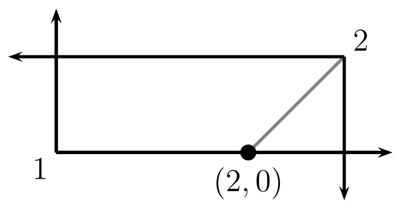
Homework #7

16.1.6 Suppose $u_1(\mathbf{x}^1) = x_1^1 + 2x_2^1$ and $u_2(\mathbf{x}^2) = \min\{x_1^2, x_2^2\}$, with endowments $\boldsymbol{\omega}^1 = (1, 0)$ and $\boldsymbol{\omega}^2 = (2, 1)$. Find the core.

Answer: $\text{MRS}_{12}^1 = 1/2$ while MRS_{12}^2 can be interpreted as anything when $x_1^2 = x_2^2$. The interior Pareto optimal allocations run from the upper right corner of the box to $(2, 0)$. The two boundary points are included. Note that $(x, 0)$ for $x < 2$ is not Pareto optimal as $(2, 0)$ is a Pareto improvement (consumer one is better off, consumer two is indifferent). The set of Pareto optima is the diagonal line in the diagram, $\{(x_1, x_2) : x_1 - 2 = x_2, x_1 \geq 2\}$.

Individual rationality requires $u_1(\mathbf{x}^1) \geq 1$ and $u_2(\mathbf{x}^2) \geq 1$. This leaves the single point $\mathbf{x}^1 = (2, 0)$ ($\mathbf{x}^2 = (1, 1)$).



16.1.8 Suppose utility is $u_i(\mathbf{x}) = \sqrt{x_1^i x_2^i}$ for $i = 1, 2, 3$ and endowments are $\boldsymbol{\omega}^1 = (1, 2)$, $\boldsymbol{\omega}^2 = (1, 3)$, and $\boldsymbol{\omega}^3 = (4, 1)$. Find the core.

Answer: Again we have identical Cobb-Douglas utility, and the Pareto set (in utility space) is $\{(u_1, u_2, u_3) \in \mathbb{R}_+^3 : u_1 + u_2 + u_3 = 6\}$. We additionally have to satisfy individual rationality: $u_1 \geq \sqrt{2}$, $u_2 \geq \sqrt{3}$ and $u_3 \geq 2$. We also must be at least as well off as in the Pareto optima for 2-consumer coalitions. Thus $u_1 + u_2 \geq \sqrt{10}$, $u_1 + u_3 \geq \sqrt{15}$ and $u_2 + u_3 \geq \sqrt{20}$.

We can simplify the conditions by substituting $u_3 = 6 - u_1 - u_2$. Then we obtain: $\sqrt{2} \leq u_1 \leq 6 - \sqrt{20}$, $\sqrt{3} \leq u_2 \leq 6 - \sqrt{15}$, and $\sqrt{10} \leq u_1 + u_2 \leq 4$ together with $u_3 = 6 - u_1 - u_2$.

Any $u_i \geq 0$ that meet the above conditions, such as $(1\frac{1}{2}, 2, 2\frac{1}{2})$ are core utility allocations. The corresponding goods allocations are $\mathbf{x}^i = u_i(1, 1)$.

20.2.1 Suppose a consumer has time additive separable preferences specified by a continuous and increasing felicity function with $u(0) = 0$ and discount factor δ, $0 < \delta < 1$. At the end of each time period, $t = 0, 1, 2, \ldots$, a constant fraction $(1 - \lambda)$ of the currently living consumers die. Death comes randomly and $0 < \lambda < 1$. Each consumer utility from consumption when alive. Death is treated as a zero-consumption/zero-utility state.

a) What is the probability that a consumer survives to period t? What is the probability p_t that a consumer dies at the end of period t? Verify that (p_t) is a probability distribution, that $\sum_{t=0}^{\infty} p_t = 1$.

b) Suppose that if alive, a consumer consumes c_t at time t. What is the expected utility from the consumption stream $\mathbf{c} = (c_0, c_1, \dots)$?

c) What is the effective discount factor for this consumer?

Answer:

a) If you are alive at any time $(t-1)$, the probability of living to time t is λ. It follows that if you are alive at time 0, the probability of being alive at time t is λ^t. The probability p_t of dying at the end of period t is then $(1-\lambda)\lambda^t$. Note that $p_t \geq 0$ and that $\sum_{t=0}^{\infty} p_t = (1-\lambda)\sum_{t=0}^{\infty} \lambda^t = (1-\lambda)(1-\lambda)^{-1} = 1$. This shows that (p_t) is a probability distribution.

b) Expected utility is $\sum_{t=0}^{\infty}(\delta\lambda)^t u(c_t)$

c) The effective discount factor is $\delta\lambda$.

20.2.4 Suppose a consumer has discount factor $0 < \delta < 1$ and period utility function $u(c) = \ln c$. The consumer has wealth $W > 0$ and faces prices $p_t = p > 0$ for all times t. Find the optimal consumption path.

Answer: Since the marginal utility of zero consumption is infinite, consumption will always be positive (unless wealth is zero). The first-order conditions are $\delta u'(c_{t+1})/u'(c_t) = p_{t+1}/p_t$. This becomes $\delta c_t/c_{t+1} = p/p = 1$, so $c_{t+1} = \delta c_t$. It follows that $c_t = \delta^t c_0$. The budget constraint is $W = \sum_t pc_t = \sum_t p\delta^t c_0 = pc_0/(1-\delta)$. Thus $c_0 = (1-\delta)W/p$ and $c_t = (1-\delta)\delta^t W/p$.

If you don't recall how to sum the infinite series, let $S = \sum_{t=0}^{\infty} \delta^t$. Then $1 + \delta S = \delta^0 + \sum_{t=1}^{\infty} \delta^t = S$. It follows that $S = (1-\delta)^{-1}$. This requires $|\delta| < 1$ for the summation to converge.

20.2.6 Suppose a consumer has discount factor $0 < \delta < 1$ and period utility function $u(c) = (1+c)^{1/2}$. The consumer has wealth $W > 0$ and faces prices $p_t = p > 0$ for all times t.

a) Use the Kuhn-Tucker Theorem to show that if $c_t = 0$ on the optimal path, then $c_{t+1} = 0$.

b) Show that there is a T with $c_t = 0$ for $t > T$.

Answer:

a) The Lagrangian is $\mathcal{L} = \sum \delta^t (1+c_t)^{1/2} - \lambda(\sum_t pc_t - W) + \sum_t \mu_t c_t$ and the first-order conditions are $(\delta^t/2)(1+c_t)^{-1/2} + \mu_t = \lambda p$. If $c_t = 0$, we have $\delta^t = \lambda p - \mu_t \leq \lambda p$, while if $c_t > 0$, $\delta^t = \lambda p(1+c_t)^{1/2} > \lambda p$. It follows that if $c_t = 0$, then $\delta^{t+1} < \delta^t \leq \lambda p$,, implying that $c_{t+1} = 0$ also.

b) If $c_t > 0$ for all t, then $\delta^t = \lambda p\sqrt{1+c_t} \geq \lambda p$ for all t. This implies $\lambda = 0$, which immediately contradicts the first-order conditions. Thus $c_t = 0$ for some t, and by part (a), $c_s = 0$ for all $s > t$. Thus there is a T with $c_t = 0$ for $t > T$.