Homework #8

20.3.3 Consider the one-sector production technology described by a production function $f(k) = k^\gamma$ for $0 < \gamma < 1$. Suppose prices are $p_t = p(1+r)^{-t}$ where $r > 0$ is the interest rate. Find k_t for every t.

Answer: The firm seeks to maximize $p_{t+1}f(k_t) - p_t k_t = p(1+r)^{-t-1}[k_t^\gamma - (1+r)k_t]$. The solution obeys the first-order condition $\gamma k_t^{\gamma-1} = 1+r$. It follows that $k_t = [(1+r)/\gamma)]^{1/(\gamma-1)}$ for all t.

20.5.1 Consider the Ramsey (one-sector) model of capital accumulation with production function $f(a) = 2a$ and felicity function $u(c) = \ln c$. The discount factor δ obeys $0 < \delta < 1$. Let $x > 0$ be the initial stock (there is no other endowment).

a) Find the optimal paths of consumption and capital accumulation.

b) Find the corresponding price path p_t.

c) The interest rate at time t is given by $1 + r_t = p_t/p_{t+1}$. Find r_t. Then compare it to the growth rate of consumption and of capital.

d) Find the value of the optimal consumption path and the present-value of the firm's profit.

Answer:

a) We start with the first-order equations: $2\delta u'(c_{t+1}) = u'(c_t)$, which imply $c_{t+1} = 2\delta c_t$. It follows that $c_t = (2\delta)^t c_0$.

By induction we find $a_t = 2^t(x - c_0) - 2^{t-1}c_1 - \cdots - c_t = 2^t[x - c_0 - \delta c_0 - \cdots - \delta^t c_0]$. Now $\sum_{s=0}^t \delta^s = (1-\delta^{t+1})/(1-\delta)$. It follows that $a_t = 2^t[x - (1-\delta^{t+1})c_0]/(1-\delta)$. We want to make c_0 as large as possible, subject to the constraint $a_t \geq 0$. The constraint can be written $x - (1-\delta^{t+1})c_0/(1-\delta) \geq 0$. In other words, $x(1-\delta) \geq (1-\delta^{t+1})c_0$ for all t. As the right-hand side is increasing, we let $t \to \infty$ to find the constraint $x(1-\delta) \geq c_0$. The maximum feasible c_0 is $c_0 = (1-\delta)x$.

Substituting back in, we obtain $c_t = (2\delta)^t(1-\delta)x$ and $a_t = (2\delta)^t \delta x$.

b) Any prices proportional to $\delta^t u'(c_t) = \delta^t/c_t = 2^{-t}/[(1-\delta)x]$ will do. In particular, $p_t = 2^{-t}$ will do. Note that the transversality condition is satisfied since $p_t a_t = \delta^{t+1}x \to 0$.

c) Now $p_t/p_{t+1} = 2^{-t}/2^{-t-1} = 2$, so $r_t = 1$. The interest rate in every time period is 100% ($r = 1$). Both consumption and capital have growth factor 2δ. The growth rate is $g = 2\delta - 1 < 2 - 1 = r$.

d) The firm's profit from production at time t is $p_{t+1}f(a_t) - p_t a_t = 2^{-t-1}2a_t - 2^{-t}a_t = 0$.

The firm's total profit is also zero.

The consumption path has value $\sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} \delta^t(1-\delta)x = (1-\delta)^{-1}(1-\delta)x = x$.

21.1.2 Consider a pure exchange economy with two goods and two states. Endowments are $\boldsymbol{\omega}^1 = \big((2,0),(0,2)\big)$ and $\boldsymbol{\omega}^2 = \big((0,2),(2,0)\big)$. Preferences are described by the utility functions $u_i(\mathbf{x}^i) = \frac{3}{8}u^i(x_{11}^i, x_{21}^i) + \frac{5}{8}u^i(x_{12}^i, x_{22}^i)$ and $u_2(\mathbf{x}^i) = \frac{1}{2}u^i(x_{11}^i, x_{21}^i) + \frac{1}{2}u^i(x_{12}^i, x_{22}^i)$ where $u^1(x_1,x_2) = \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$ and $u^2(x_1,x_2) = \frac{1}{2}\sqrt{x_1} + \frac{1}{2}\sqrt{x_2}$. Note that the consumers have different beliefs about the probabilities of the two states. Consumer one thinks that state two is more likely, consumer two thinks both states are equally likely.

a) Find the Arrow-Debreu equilibrium

b) Are the consumers fully insured?

Answer:

a) Consumer one has Cobb-Douglas preferences with weights 3/16 on each good in state one and 5/16 on each good in state two. Consumer one's demand is

$$\mathbf{x}^1(\mathbf{p}) = \frac{2p_{11} + 2p_{22}}{16}\left(\frac{3}{p_{11}}, \frac{3}{p_{21}}, \frac{5}{p_{12}}, \frac{5}{p_{22}}\right).$$

Consumer two has Lagrangian

$$\mathcal{L} = \sum_{\ell,s} \frac{1}{4}\sqrt{x_{\ell,s}} + \lambda\left(\mathbf{p}\cdot\boldsymbol{\omega}^2 - \sum_{\ell,s} p_{\ell,s}x_{\ell,s}\right).$$

The first-order conditions are

$$\sqrt{x_{\ell,s}} = \frac{1}{8\lambda p_{\ell,s}},$$

so $x_{\ell,s} = 1/64\lambda^2 p_{\ell,s}^2$. The budget constraint then yields $\mathbf{p}\cdot\boldsymbol{\omega}^2 = (1/64\lambda^2)\sum 1/p_{\ell,s}$, so

$$64\lambda^2 = \frac{\sum 1/p_{\ell,s}}{2p_{21} + 2p_{12}}.$$

Consumer two's demand is

$$\mathbf{x}^2(\mathbf{p}) = \frac{2p_{21} + 2p_{12}}{\sum 1/p_{\ell,s}}\left(\frac{1}{p_{11}^2}, \frac{1}{p_{21}^2}, \frac{1}{p_{12}^2}, \frac{1}{p_{22}^2}\right).$$

We now set supply equal to demand in each market.

$$2 = \frac{6p_{11} + 6p_{22}}{16p_{\ell,1}} + \frac{2p_{21} + 2p_{12}}{p_{\ell,1}^2 \sum 1/p_{\ell,s}}$$

and

$$2 = \frac{10p_{11} + 10p_{22}}{16p_{\ell,2}} + \frac{2p_{21} + 2p_{12}}{p_{\ell,2}^2 \sum 1/p_{\ell,s}}.$$

It follows from the first equation that both goods have the same price in state one $p_1 = p_{11} = p_{21}$ while the second equation tells us that they have possibility different price in state two $p_2 = p_{12} = p_{22}$. Plugging this in the equilibrium conditions yields $13p_1 = 11p_2$. We normalize so that $p_1 = 11$, so $p_2 = 13$. Both consumers have income 48. Consumer one has demand

$$\mathbf{x}^1 = \left(\left(\frac{9}{11}, \frac{9}{11} \right), \left(\frac{15}{13}, \frac{15}{13} \right) \right)$$

and consumer two's demand is

$$\mathbf{x}^2 = \left(\left(\frac{13}{11}, \frac{13}{11} \right), \left(\frac{11}{13}, \frac{11}{13} \right) \right).$$

b) Since each consumer consumes the different amounts in both states, neither consumer is fully insured.

21.2.3 Consider an exchange economy with 2 consumers, 1 good, and 3 states of the world. Let x_s^i denote consumer i's consumption of the one good in state s. Each consumer has utility function

$$u(\mathbf{x}^i) = \sum_{s=1}^{3} \ln x_s^i.$$

The endowments are $\boldsymbol{\omega}^1 = (1, 2, 3)$ and $\boldsymbol{\omega}^2 = (3, 2, 1)$. Find the Arrovian securities equilibrium. Be sure to indicate the spot market prices, prices of the securities, and the equilibrium consumption by each consumer.

Hint: Since we can normalize each spot market separately, we can set $p_s = 1$ for each s.

Answer: With $p_s = 1$ for every s, both consumers consume their income in state s. I.e., $x_{1s}^i = \boldsymbol{\omega}_{1s}^i + z_s^i$. Thus indirect utility is $v^i(\mathbf{z}) = \sum_{s=1}^{3} \ln(\boldsymbol{\omega}_{1s}^i + z_s^i)$. The first-order conditions are then $\lambda q_s = 1/(\boldsymbol{\omega}_{1s}^i + z_s^i)$ or $\boldsymbol{\omega}_{1s}^i + z_s^i = 1/(\lambda q_s)$. Summing over i and using asset market clearing, we find $4 = 2/(\lambda q_s)$. The q_s must be equal, so we may set $q_s = 1$ for every security s. Now $\lambda = 1/2$. Substituting back in the first-order conditions, we find $2 = \boldsymbol{\omega}^i_{1s} = z^i_s$. Thus $\mathbf{z}^1 = (1, 0, -1)$ and $\mathbf{z}^2 = (-1, 0, 1)$. The corresponding goods allocations are $\mathbf{x}^1 = (2, 2, 2)$ and $\mathbf{x}^2 = (2, 2, 2)$.