Micro I Midterm, February 28, 2017

1. Suppose $e(\mathbf{p}, \bar{u}) = p_1 p_2 \bar{u}^2/(p_1 + p_2)$ is a consumer's expenditure function.

   a) Find the Hicksian (compensated) demand functions.

   Answer: The Shepard-McKenzie Lemma tells us that $\mathbf{h}(\mathbf{p}, \bar{u}) = d_{\mathbf{p}} e(\mathbf{p}, \bar{u})$. Thus

   $$\begin{pmatrix} h_1(\mathbf{p}, \bar{u}) \\ h_2(\mathbf{p}, \bar{u}) \end{pmatrix} = \frac{\bar{u}^2}{(p_1 + p_2)^2} \begin{pmatrix} p_2^2 \\ p_1^2 \end{pmatrix}.$$

   b) Find the direct utility function $u(\mathbf{x})$.

   Answer: We first write the Hicksian demands in terms of $\bar{u}$, not $\bar{u}^2$. Then

   $$\sqrt{h_1} = \frac{p_2}{p_1 + p_2}\,\bar{u} \text{ and } \sqrt{h_2} = \frac{p_1}{p_1 + p_2}\,\bar{u}.$$

   Then $\sqrt{h_1} + \sqrt{h_2} = \bar{u}$. It follows that

   $$u(\mathbf{x}) = \sqrt{x_1} + \sqrt{x_2}.$$

   c) Find the indirect utility function $v(\mathbf{p}, m)$.

   Answer: By duality, $e\Big(\mathbf{p}, v(\mathbf{p}, m)\Big) = m$. Thus

   $$\frac{p_1 p_2}{p_1 + p_2} v^2(\mathbf{p}, m) = m.$$

   Rearranging, we find

   $$v^2(\mathbf{p}, m) = \frac{m(p_1 + p_2)}{p_1 p_2},$$

   and taking the square root yields

   $$v(\mathbf{p}, m) = \sqrt{\frac{m(p_1 + p_2)}{p_1 p_2}}.$$

2. A firm uses two inputs z_1 and z_2 to produce a single output. The production function is $f(\mathbf{z}) = -1 + \sqrt{z_1} + \sqrt{1 + z_2}$ for $z_1, z_2 \geq 0$. The output price is p and the input prices are w_1 and w_2.

   a) Find all profit-maximizing input levels when prices are (p, w_1, w_2). What are the corresponding output levels?

Answer: Since $\partial f/\partial z_1(0+, z_2) = +\infty$, we do not have to worry about the constraint on z_1. However, $\partial f/\partial z_2(z_1, 0) = 1/2$, so the constraint $z_2 \geq 0$ might bind. Accordingly, we use the Lagrangian $\mathcal{L} = -p + p\sqrt{z_1} + p\sqrt{1+z_2} - w_1 z_1 - w_2 z_2 + \mu z_2$. The positive sign on μ arises since the constraint is $z_2 \geq 0$. The objective is concave, so it suffices to check the first-order conditions:

$$w_1 = \frac{p}{2\sqrt{z_1}},$$
$$w_2 = \mu + \frac{p}{2\sqrt{1+z_2}}.$$

Now $p/2\sqrt{1+z_2} \leq p/2$, so $\mu > 0$ whenever $w_2 > p/2$. Complementary slackness then implies $z_2 = 0$. Of course, if $w_2 \leq p/2$, $\mu = 0$ and z_2 is determined by the first-order condition.

It follows that the optimal inputs are

$$z_1 = p^2/4w_1^2,$$
$$z_2 = \begin{cases} -1 + p^2/4w_2^2 & \text{when } 2w_2 \leq p \\ 0 & \text{when } 2w_2 > p. \end{cases}$$

The corresponding net output is

$$q = \begin{cases} -1 + \frac{p(w_1+w_2)}{2w_1w_2} & \text{when } 2w_2 \leq p \\ \frac{p}{2w_1} & \text{when } 2w_2 > p. \end{cases}$$

b) Are there any strictly positive price vectors (p, w_1, w_2) where profit cannot be maximized? Are there any strictly positive price vectors (p, w_1, w_2) where one of the inputs is not used?

Answer: Profit can be maximized for any strictly positive price vector (p, w_1, w_2). Input two is not used when $p < 2w_2$.

3. A firm has production set $Y = \{\mathbf{y} \in \mathbb{R}^3 : y_1, y_2 \leq 0, -\sqrt{-y_1} + y_2 + y_3 \leq 0\}$.

a) Does the technology obey constant returns to scale? Explain.

Answer: No. The net output $(-1, 0, 1) \in Y$ since $-1 + 0 + 1 \leq 0$ but $(-2, 0, 2) \notin Y$ because $-\sqrt{2} + 0 + 2 \approx 0.59 > 0$.

b) Find the supply (net output) correspondence $\mathbf{y}(\mathbf{p})$.

Answer: Profit from net output $\mathbf{y}$ is $p_1y_1 + p_2y_2 + p_3y_3$. Now $y_3 \le \sqrt{-y_1} - y_2$, so profit is at most $p_1y_1 + p_2y_2 + p_3\sqrt{-y_1} - p_3y_2 = p_1y_1 + p_3\sqrt{-y_1} + (p_2 - p_3)y_2$.

The first-order condition for y_1 is $p_1 = p_3/2\sqrt{-y_1}$, so $y_1 = -p_3^2/4p_1^2$, yielding output $p_3/2p_1$.

Now consider y_2. If $p_2 > p_3$, the fact that $y_2 \le 0$ implies $y_2 = 0$ to maximize profit. If $p_2 < p_3$, profit can be increased without bound. Finally, if $p_2 = p_3$, profit from use of good two as input will be zero at any input level.

The net output correspondence is

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \left(-\frac{p_3^2}{4p_1^2}, 0, \frac{p_3}{2p_1}\right)^{\mathsf{T}} & \text{when } p_2 > p_3 \\ \left(-\frac{p_3^2}{4p_1^2}, y_2, \frac{p_3}{2p_1} - y_2\right)^{\mathsf{T}} & \text{for any } y_2 \le 0, \text{ when } p_2 = p_3 \\ \text{no solution} & \text{when } p_2 < p_3. \end{cases}$$

c) Find the profit function $\pi(\mathbf{p})$.

Answer: Here

$$\pi(\mathbf{p}) = \begin{cases} \frac{p_3^2}{4p_1} & \text{when } p_2 \ge p_3 \\ +\infty & \text{when } p_2 < p_3. \end{cases}$$

4. Suppose there are two goods and two consumers with utility $u_1(\mathbf{x}^1) = \min\{x_1^1, x_2^1\}$ and $u_2(\mathbf{x}^2) = \min\{2x_1^2, x_2^2\}$.

a) Find the Marshallian demand for each consumer given prices $\mathbf{p} \gg \mathbf{0}$ and incomes $m_i \ge 0$ for $i = 1, 2$.

Answer: Consumer 1's demand obeys $x_1^1 = x_2^1$, so

$$\mathbf{x}^1(\mathbf{p}, m_1) = \frac{m_1}{p_1 + p_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly

$$\mathbf{x}^1(\mathbf{p}, m_1) = \frac{m_2}{p_1 + 2p_2}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

b) Now compute market demand $\mathbf{x}^M(\mathbf{p}, m_1, m_2)$.

Answer:

$$\mathbf{x}^M(\mathbf{p}, m_1, m_2) = \begin{pmatrix} \frac{m_1}{p_1+p_2} + \frac{m_2}{p_1+2p_2} \\ \frac{m_1}{p_1+p_2} + \frac{2m_2}{p_1+2p_2} \end{pmatrix}.$$

c) Find non-negative (m_1, m_2) and (m_1', m_2') with $m_1 + m_2 = m_1' + m_2'$ where $\mathbf{x}^M(\mathbf{p}, m_1, m_2) \neq \mathbf{x}^M(\mathbf{p}, m_1', m_2')$ for some $\mathbf{p} \gg \mathbf{0}$.

Answer: Many combinations will work. For example, we can take $m_1 = 1$, $m_2 = 0$, $m_1' = 0$, and $m_2' = 1$ and set $\mathbf{p} = (1, 1)^T$. Then

$$\mathbf{x}^M\Big((1,1), 1, 0\Big) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \neq \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \mathbf{x}^M\Big((1,1), 0, 1\Big).$$