Micro II Final, April 26, 2018

1. Which of the following utility functions on $\mathbb{R}^3_+$ represents an additive separable preference order? Either show how to transform utility into an additive separable form, or show that it cannot be done.

   a) $u_2(\mathbf{x}) = (1 + x_1)(1 + x_2)/(1 + x_3)$.

   b) $u_3(\mathbf{x}) = (1 + x_1 + x_2)(1 + x_2)^2(1 + x_3)$.

   Answer:

   a) The utility u_2 is also additive separable since the equivalent utility $\log u_2 = \log(1 + x_1) + \log(1 + x_2) - \log(1 + x_3)$ has the additive separable form. The lack of monotonicity is not a problem.

   b) The utility u_3 is not additive separable. Consider the marginal rate of substitution

   $$\begin{aligned} \mathrm{MRS}_{23} &= \frac{(1 + x_2)^2(1 + x_3) + 2(1 + x_1 + x_2)(1 + x_2)(1 + x_3)}{(1 + x_1 + x_2)(1 + x_2)^2} \\ &= \frac{(3 + 2x_1 + 3x_2)}{(1 + x_1 + x_2)(1 + x_2)}(1 + x_3) \end{aligned}$$

   Since this marginal rate of substitution depends on a good other than good two or three, the associated preference order is not separable with respect to the partition $\Big\{\{1\}, \{2, 3\}\Big\}$.

   A similiar argument using MRS_{13} (but not MRS_{12}) comes to the same conclusion.

2. Consider the following Ramsey problem. Suppose a consumer has utility $\sum_{t=0}^{\infty} \delta^t u(c_t)$ where $0 < \delta < 1$ and the felicity function is $u(c) = 1/c$. The production function is $f(a) = \beta a$ where $\beta > 1$. Suppose that there is an optimal path with $c_t > 0$ for every t.

   a) Does consumption grow? If so, what is the growth factor.

   b) Is the (consumption) transversality condition satisified?

   Answer:

   a) The Euler equations are

   $$\delta f'(a_t)u'(c_{t+1}) = u'(c_t)$$

   yielding

   $$\delta\beta/c_{t+1}^2 = 1/c_t^2.$$

It follows that $c_{t+1} = (\delta\beta)^{1/2}c_t$, implying that $c_t = (\delta\beta)^{t/2}c_0$.

Consumption grows by the growth factor $(\delta\beta)^{1/2}$ when $\delta\beta > 1$, is constant if $\delta\beta = 1$, and shrinks if $\delta\beta < 1$, all of which are possible.

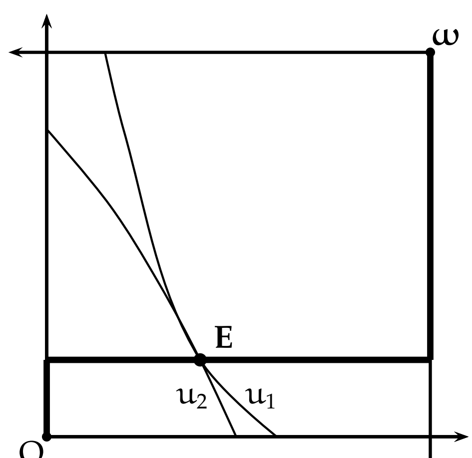
b) The consumption transversality condition is that $p_t c_t \to 0$ where $p_t = \delta^t u'(c_t) = \delta^t/c_t^2$. Thus $p_t c_t = \delta^t/c_t$. Now $c_t = (\delta\beta)^{t/2}c_0$, so $p_t c_t = (\delta/\beta)^{t/2}/c_0$. Since $\delta < 1$ and $\beta > 1$, $\delta/\beta < 1$, implying that the transversality condition is satisfied.

3. An exchange economy has two consumers with utility functions $u_1(\mathbf{x}^1) = x_1^1 + \sqrt{x_2^1}$ and $u_2(\mathbf{x}^2) = x_1^2 + 2\sqrt{x_2^2}$. Endowments are $\boldsymbol{\omega}^1 = (2,1)$ and $\boldsymbol{\omega}^2 = (3,4)$. Find the core.

Answer: Since there are two consumers, the core consists of allocations that are both Pareto optimal and individually rational. Interior Pareto optima must have the same marginal rate of substitution for both consumers. Thus $MRS_{12}^1 = 2\sqrt{x_2^1} = MRS_{12}^2 = \sqrt{x_2^2}$, implying $4x_2^1 = x_2^2$. Then $x_2 = x_2^1 + x_2^2 = 5x_2^1 = \omega_2^1 = 5$. It follows that $x_2^1 = 1$ and $x_2^2 = 4$ at any interior Pareto optimum. The rest of the Pareto optima are the allocations $\big((0, x_2^1), (5, 5 - x_2^1)\big)$ with $0 \le x_2^1 \le 1$ and $\big((5, x_2^1), (0, 5 - x_2^1)\big)$ with $1 \le x_2^1 \le 4$. They are illustrated by the heavy line in the diagram.

Now the endowment point **E** is itself Pareto optimal because $MRS_{12}^1 = 2 = MRS_{12}^2$ as illustrated in the diagram. Any other individually rational allocation must Pareto improve on **E**, which is impossible. Thus **E** is the only point in the core.

The situation is illustrated in the diagram.



4. Suppose two consumers have identical convex utility functions on $\mathbb{R}_+^2$, $u_i(\mathbf{x}) = \max\{x_1, x_2\}$. Endowments are $\boldsymbol{\omega}^1 = (3,0)$ and $\boldsymbol{\omega}^2 = (0,3)$. Find **all** equilibria for this exchange economy in $\mathbb{R}_+^2$, or show that there are no equilibria.

Answer: Preferences are strictly monotonic, so any equilibrium prices have to be strictly positive. We take good one as numéraire and write $\mathbf{p} = (1, p)$. Incomes are then $m^1 = 3$ and $m^2 = p$. If the prices of the two goods are different, each consumer will only demand the cheapest one. Thus if $p < 1$ market demand is $(3 + 3p, 0)$ and if $p > 1$ demand is $(0, 3 + 3/p)$. In either case, there is excess demand, so the only possible equilibrium price is $p = 1$. Due to the convex utility functions, demand for each consumer is either $(3, 0)$ or $(0, 3)$. It follows that there are two possible equilibria, both with prices $\mathbf{p} = (1, 1)$ but with different individual consumption vectors. The two equilibrium allocations are $(\mathbf{x}^1, \mathbf{x}^2) = \big((3, 0), (0, 3)\big)$ and $(\mathbf{x}^1, \mathbf{x}^2) = \big((0, 3), (3, 0)\big)$.

5. Consider a production economy with 2 goods and 2 consumers. There is one firm with technology set $Y = \{(y_1, y_2) : y_1 \le 0, y_2 \le -y_1/2\}$. Both consumers have the same utility function, $u(\mathbf{x}) = \min\{x_1, x_2\}$. Endowments are $\boldsymbol{\omega}^1 = (2, 2)$ and $\boldsymbol{\omega}^2 = (4, 0)$. Find all competitive equilibria $(\mathbf{x}^i, \mathbf{y}, \mathbf{p})$.

   Answer: We first consider whether the firm operates. If it does not, there is excess supply of good one, and its price must be zero. But then it is profitable to operate the firm. It follows that the firm must operate in equilibrium.

   Since the firm oprerates, its profit must be zero. The firm's profit is $p_1y_1 + p_2y_2$ which is maximized at $y_2 = -y_1/2$. Thus profit is $(p_1 - p_2/2)y_1$, which must be zero due to constant returns to scale. It follows that $p_1 = p_2/2$. We may take good one as numéraire, yielding prices $\mathbf{p} = (1, 2)$ (or any positive multiple thereof).

   Since both prices are positive, there is no excess demand, which means that the supply of both goods must be equal: $\omega_1 + y_1 = \omega_2 - y_1/2$. In other words, $6 + y_1 = 2 - y_1/2$. Thus $-4 = 3y_1/2$, so $y_1 = -8/3$. The net output vector is $\mathbf{y} = (-8/3, 4/3)$. Total supply is $\boldsymbol{\omega} + \mathbf{y} = (10/3, 10/3)$.

   Consumer incomes are $m^i = \mathbf{p}\cdot\boldsymbol{\omega}^i$, so $m^1 = 6$ and $m^2 = 4$. It follows that consumer one will consume 60% of the goods supplied and consumer two will consumer 40%. Thus $\mathbf{x}^1 = (2, 2)$ and $\mathbf{x}^2 = (4/3, 4/3)$.

   The equilibrium is $\mathbf{p} = (1, 2)$, $\mathbf{y} = (-8/3, 4/3)$, $\mathbf{x}^1(2, 2)$, and $\mathbf{x}^2 = (4/3, 4/3)$. This is the only equilibrium allocation, and any positive multiple of $\mathbf{p}$ yields the same equilibrium allocation.