Homework #1

2.2.7 Suppose $\succsim$ is a preference order on $\mathfrak{X}$ and let $\succ$ be the derived strict preference relation. Show that $\succ$ is *irreflexive*, that is, there is no $\mathbf{x} \in \mathfrak{X}$ with $\mathbf{x} \succ \mathbf{x}$.

Answer: If $\mathbf{x} \succ \mathbf{y}$ is true, then by the definition of $\succ$, both $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \not\succsim \mathbf{x}$. Applying this to the case where $\mathbf{y} = \mathbf{x}$, we we find that $\mathbf{x} \succsim \mathbf{x}$ is both true and false! This cannot be, so no $\mathbf{x}$ obeys $\mathbf{x} \succ \mathbf{x}$.

2.3.6 Find the marginal rates of substitution for the quasi-linear function $u(\mathbf{x}) = ax_1 + \varphi(x_2)$ where φ is a $\mathcal{C}^1$ function.

Answer: The only marginal rate of substitution to compute is $\mathrm{MRS}_{12} = a/\varphi'(x_2)$.

2.3.8 Define u and v on $\mathfrak{X} = \mathbb{R}^3_+$ by $u(\mathbf{x}) = (x_1x_2x_3)^{1/3}$ and $v(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2$. Are u and v equivalent utility functions? Justify your answer.

Answer: Consider $\mathrm{MRS}^u_{12} = x_2/x_1$ and $\mathrm{MRS}^v_{12} = x_1/x_2$. Since the marginal rates of substitution differ for $x_1 \neq x_2$, the utility functions cannot be equivalent.

3.2.2 Let $u\colon \mathbb{R}_+ \to \mathbb{R}$ be defined by $u(x) = x$ for $x < 1$ and $u(x) = x + 1$ for $x \geq 1$. Show that the preference order defined by u is a continuous preference order.

Answer: There are at least two ways to solve this problem. One is to proceed from the definition of continuity. To do that, we must show that both the upper and lower contour sets are closed. Now

$$\{x : u(x) \geq u^0\} = \begin{cases} [u^0 - 1, +\infty) & \text{if } u^0 \geq 2 \\ [1, +\infty) & \text{if } 1 \leq u^0 \leq 2 \\ [u^0, +\infty) & \text{if } 0 \leq u^0 \leq 1 \end{cases}$$

so the upper contour set is closed. Similarly,

$$\{x : u(x) \leq u^1\} = \begin{cases} [0, u^1 - 1] & \text{if } u^1 \geq 2 \\ [0, 1] & \text{if } 1 \leq u^1 \leq 2 \\ [0, u^1] & \text{if } 0 \leq u^1 \leq 1 \end{cases}$$

so the lower contour set is closed. It follows that the preference order defined by u is continuous.

A second way to solve the problem is to use the fact that $u(x) \geq u(y)$ if and only if $x \geq y$. The preference order is the same as that obtained from the utility function $v(x) = x$. Since v is continuous, it represents a continuous preference order, the same preference order defined by u.

3.4.1 Suppose u is a $\mathcal{C}^1$ function on $\mathbb{R}^2_{++}$ with $du \neq \mathbf{0}$ on $\mathbb{R}^2_{++}$. Show that u is locally non-satiated.

Answer: Let $\mathbf{x} \in \mathbb{R}^2_{++}$. Choose k with $\partial u/\partial x_k \neq 0$. This is possible since $du \neq \mathbf{0}$. Choose ϵ small with $\epsilon > 0$ if $\partial u/\partial x_k(\mathbf{x}) > 0$ and $\epsilon < 0$ if $\partial u/\partial x_k(\mathbf{x}) < 0$. The mean value theorem tells us that

$$u(\mathbf{x} + \epsilon\mathbf{e}^k/2) = u(\mathbf{x}) + \frac{\epsilon}{2}\frac{\partial u}{\partial x_k}$$

where $\partial u/\partial x_k$ is evaluated at a point between $\mathbf{x}$ and $\mathbf{x} + \epsilon\mathbf{e}^k/2$. By choosing ϵ small enough, we guarantee that $\epsilon\partial u/\partial x_k > 0$. It follows that $u(\mathbf{x} + \epsilon\mathbf{e}^k/2) > u(\mathbf{x})$, showing that u is locally non-satiated.