Homework #2

3.3.1 Consider the Cobb-Douglas utility $u(\mathbf{x}) = x_1^\alpha x_2^\beta$ where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Use the Hessian to show that u is a strictly concave function for $x_1, x_2 > 0$.

Answer: We compute the Hessian matrix

$$\mathbf{H} = \begin{pmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^\beta & \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} \\ \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} & \beta(\beta-1)x_1^\alpha x_2^{\beta-2} \end{pmatrix}.$$

Now $0 < \alpha < 1$, so the first leading principal minor $H_1 = \alpha(\alpha-1)x_1^{\alpha-2}x_2^\beta < 0$ and the second leading principal minor $H_2 = [\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2]x_1^{2\alpha-2}x_2^{2\beta-2} = \alpha\beta(1-\alpha-\beta)x_1^{2\alpha-2}x_2^{2\beta-2} > 0$. This shows that H is negative definite, implying that u is strictly concave.

3.5.3 Show that $u(x,y) = (1+x)(1+y)+y^{1/2}$ does not have an additive separable representation.

Answer: Suppose there is a φ so that $v = \varphi \circ u$ is additive separable. We now compute $\partial v/\partial x = (1+y)\varphi(u)$ and $\partial^2 v/\partial x \partial y = \varphi'(u) + (1+y)[1+x+\frac{1}{2}y^{-1/2}]\varphi''(u) = \varphi'(u) + [u - \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}]\varphi''(u)$. For v to be additive separable, we must have $0 = \varphi'(u) + [u - \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}]\varphi''(u)$. However, the presence of y indicates that φ is not solely a function of u. Thus it is impossible to find a monotonic function of u that yields the required condition ($\partial^2 v/\partial x\partial y = 0$).

3.5.7 Let u be a continuous utility function and $\mathcal{P}$ be a partition of goods. Suppose there is a continuous function v that is increasing in each argument and continuous subutility functions u_P defined on each $\mathbf{x}_P$ with $u(\mathbf{x}) = v\big[\big(u_P(\mathbf{x}_P)\big)_{P\in\mathcal{P}}\big]$. Show that u is weakly separable relative to $\mathcal{P}$.

Answer: We must show that u induces an order on each commodity group $P \in \mathcal{P}$. Let $\mathcal{P} = \{P_i\}_{i=1}^K$. Now suppose $\mathbf{x} = (\mathbf{x}_{P_1}, \mathbf{x}_{P_1^c}) \succsim \mathbf{x} = (\mathbf{y}_{P_1}, \mathbf{x}_{P_1^c})$. Let $u_i = u_{P_i}(\mathbf{x}_{P_i})$. We can then write this preference in utility terms as

$$v(u_1, \dots, u_K) \geq v(u_{P_1}(\mathbf{y}_{P_1}), u_2, \dots, u_K)$$

Since v is increasing in argument 1, this is equivalent to $u_1 = u_{P_1}(\mathbf{x}_{P_1}) \geq u_{P_1}(\mathbf{y}_{P_1})$. The actual values of $u_2, \dots, u_K$ don't matter here, so v induces an order on P_1. In fact, $\mathbf{x} \succsim_{P_1} \mathbf{y}$ if and only if $u_1 = u_{P_1}(\mathbf{x}_{P_1}) \geq u_{P_1}(\mathbf{y}_{P_1})$.

By repeating for every P_i, we find that v induces an order on each P_i. In other words, it is weakly separable relative to $\mathcal{P} = \{P_i\}$.

4.3.1 Let $L = 2$ and let utility have the quasi-linear form $u(\mathbf{x}) = x_1 + \ln x_2$.

a) Find the Marshallian demand and the indirect utility function.

b) For which prices are both goods consumed?

c) For fixed p_2 and m, sketch x_1 as a function of p_1.

d) For fixed p_1 and m, sketch x_2 as a function of p_2.

Answer:

a) The Lagrangian is $\mathcal{L} = x_1 + \ln x_2 + \lambda(m - p_1x_1 - p_2x_2) + \mu_1x_1 + \mu_2x_2$. This yields first-order conditions $1+\mu_1 = \lambda p_1$ and $1/x_2+\mu_2 = \lambda p_2$. The second condition cannot be satisfied if $x_2 = 0$, so we conclude $x_2 > 0$. Complementary slackness then implies $\lambda = 1/p_2x_2$. Since $\lambda > 0$, complementary slackness also implies $p_1x_1 + p_2x_2 = m$.

There are now two possibilities: $x_1 = 0$ and $x_1 > 0$. We take $x_1 = 0$ first. Here $p_2x_2 = m$, so $\lambda = 1/m$. In order to satisfy the remaining first-order condition, we must have $p_1 \geq m$. In that case $x_1 = 0$ and $x_2 = m/p_2$ is our solution. Indirect utility is then $v(\mathbf{p}, m) = \ln m/p_2$.

Now consider $x_1 > 0$. Complementary slackness gives $\mu_1 = 0$, so $\lambda p_1 = 1$. The $p_1 = 1/\lambda = p_2x_2$. The budget constraint then tells us that $p_1x_1 = m - p_2x_2 = m - p_1$. This only makes sense if $m \geq p_1$. In that case $x_1 = (m - p_1)/p_1$ and $x_2 = p_1/p_2$ and $v(\mathbf{p}, m) = (m - p_1)/p_1 + \ln p_1/p_2$.

Note that the two solution coincide when $m = p_1$.

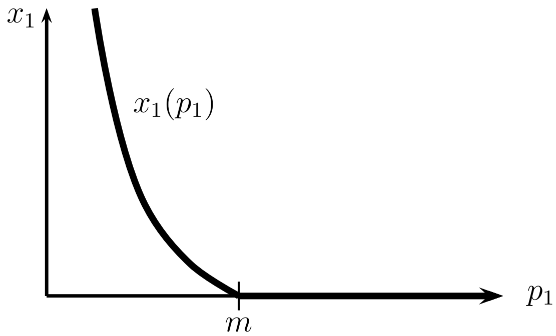
The demand function is:

$$\mathbf{x}(\mathbf{p}, m) = \begin{cases} \left(\frac{m-p_1}{p_1}, \frac{p_1}{p_2}\right) & \text{when } p_1 \leq m \\ \left(0, \frac{m}{p_2}\right) & \text{when } p_1 \geq m. \end{cases}$$

b) Using the solution above, we find that both goods are consumed for prices with $p_1 < m$.

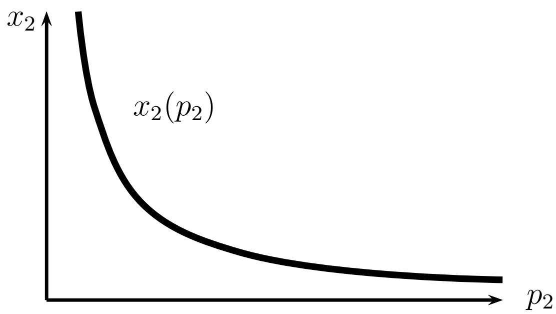
c)

Demand for good one.



d)

Demand for good two is either p_1/p_2 or m/p_2, both of which look similar.



4.5.6 Let u be a homogeneous of degree one utility function on $\mathbb{R}^L_+$. Does indirect utility have the Gorman form?

Answer: Yes, it has the Gorman form. Since u is homothetic, $\mathbf{x}(\mathbf{p},m) = m\mathbf{x}(\mathbf{p},1)$. Now $v(\mathbf{p},m) = u\big(\mathbf{x}(\mathbf{p},m)\big) = u\big(m\mathbf{x}(\mathbf{p},1)\big) = mu\big(\mathbf{x}(\mathbf{p},1)\big)$. Setting $b(\mathbf{p}) = u\big(\mathbf{x}(\mathbf{p},1)\big)$, we find $v(\mathbf{p},m) = mb(\mathbf{p})$, which is in Gorman form with $a(\mathbf{p}) = 0$.