Homework #3

5.1.4 Suppose utility is given by an arbitrary linear function on $\mathbb{R}^L_+$. Find the expenditure function and compensated demands. For the values of $\mathbf{p}$ where $\mathbf{h}(\mathbf{p}, \bar{u})$ is a function, determine which goods are substitutes and which are complements, if any.

Answer: Let $u(\mathbf{x}) = \mathbf{a}{\cdot}\mathbf{x}$ for some $\mathbf{a} > \mathbf{0}$. Only the goods ℓ with $a_\ell/p_\ell = \max_k\{a_k/p_k\}$ will be consumed. If there are two such goods, there will be many solutions. However, they will all result in the same expenditure.

Choose ℓ with $a_\ell/p_\ell = \max_k\{a_k/p_k\}$. We may assume that only good ℓ is consumed, so $x_\ell = \bar{u}/a_\ell$. It follows that expenditure is $\bar{u}p_\ell/a_\ell = \bar{u}/\max_k\{a_k/p_k\}$. The last is true regardless of which of the best goods are consumed, so expenditure can be written

$$e(\mathbf{p}, \bar{u}) = \bar{u}\min_k\{p_k/a_k\}.$$

If $p_\ell/a_\ell < p_k/a_k$ for $k \neq \ell$, the Hicksian demand for ℓ is $h_\ell(\mathbf{p}, \bar{u}) = \bar{u}/a_\ell$ and $h_k(\mathbf{p}, \bar{u}) = 0$ for $k \neq \ell$. If there are two or more minimizers of p_k/a_k, those goods may have positive Hicksian demand with $\sum_\ell a_\ell h_\ell = \bar{u}$ while demand for the other goods is zero.

Unless there are two or more minimizers of p_k/a_k, the derivatives of the Hicksian demands are zero, indicating that the goods are neither substitutes nor complements.

If there are multiple goods with positive Hicksian demand, any decrease in the price of one will cause demand for the others to fall to zero, while an increase in price will increase demand for at least some of the others. Demand for other (non-tied) goods will remain at zero. The goods with $p_\ell/a_\ell = \min\{p_k/a_k\}$ are substitutes for each other, and the other goods are neither substitutes nor complements.

5.1.7 Suppose the utility function $u(\mathbf{x}) = x_1 + \sqrt{x_2x_3}$. Find the expenditure function, compensated demands, and determine which goods are substitutes and which are complements.

Answer: The Lagrangian is $\mathbf{p}{\cdot}\mathbf{x} - \lambda(x_1 + \sqrt{x_2x_3} - \bar{u}) - \mu_1x_1 - \mu_2x_2 - \mu_3x_3$. The first-order conditions are $p_1 = \lambda + \mu_1$, $p_2 = (\lambda/2)\sqrt{x_3/x_2} + \mu_2$, and $p_3 = (\lambda/2)\sqrt{x_2/x_3} + \mu_3$.

If $x_1, x_2, x_3 > 0$, then $p_1 = \lambda$, $2p_2 = \lambda\sqrt{x_3/x_2}$ and $2p_3 = \lambda\sqrt{x_2/x_3}$. As usual, we eliminate λ to find $p_2x_2 = p_3x_3$. This implies $\lambda = 2\sqrt{p_2p_3}$. The first condition then yields $p_1 = 2\sqrt{p_2p_3}$. In this case any $\mathbf{x} \geq \mathbf{0}$ with $x_1 + x_2\sqrt{p_2/p_3} = \bar{u}$ and $x_3 = p_2x_2/p_3$ minimizes expenditure. The minimum is $p_1\bar{u} = 2\bar{u}\sqrt{p_2p_3}$.

If $x_2 = 0$ or $x_3 = 0$ we must have $x_2 = x_3 = 0$ for the first-order conditions to make sense. Then $x_1 = \bar{u}$, $p_1 = \lambda$, and $e(\mathbf{p}, \bar{u}) = p_1\bar{u}$. However we interpret x_2/x_3, multiplying the last two first-order conditions yields $p_2p_3 \geq \lambda/4$. In other words, it requires $2\sqrt{p_2p_3} \geq p_1$.

If $x_1 = 0$, both x_2 and x_3 must be positive. As before, $p_2x_2 = p_3x_3$. It follows that $\bar{u} = x_2\sqrt{p_2/p_3}$, so $x_2 = \bar{u}\sqrt{p_3/p_2}$ and $x_3 = \bar{u}\sqrt{p_2/p_3}$. Expenditure is $e(\mathbf{p}, \bar{u}) = 2\bar{u}\sqrt{p_2p_3}$. Finally, the first-order conditions require $2\sqrt{p_2p_3} \leq p_1$.

Summing up, the Hicksian demands are:

$$\mathbf{h}(\mathbf{p}, \bar{u}) = \begin{cases} (\bar{u}, 0, 0) & \text{if } p_1 < 2\sqrt{p_2p_3} \\ \{\mathbf{x} \geq \mathbf{0} : x_1 + x_2\sqrt{p_2p_3} = \bar{u}, p_2x_2 = p_3x_3\} & \text{if } p_1 = 2\sqrt{p_2p_3} \\ (0, \bar{u}\sqrt{p_3/p_2}, \bar{u}\sqrt{p_2/p_3}) & \text{if } p_1 > 2\sqrt{p_2p_3} \end{cases}$$

and the expenditure function is

$$e(\mathbf{p}, \bar{u}) = \bar{u}\min\{p_1, 2\sqrt{p_2p_3}\}$$

When $p_1 < 2\sqrt{p_2p_3}$, all of the $\partial h_k/\partial p_\ell = 0$, so goods are neither substitutes nor complements. When $p_1 > 2\sqrt{p_2p_3}$, $\partial h_1/\partial p_k = 0$ for $k = 1, 2$ and $\partial h_2/\partial p_3 = \partial h_3/\partial p_2 = \bar{u}/(2\sqrt{p_2p_3}) > 0$. Thus goods 2 and 3 are substitutes. The others are not related.

5.3.2 Consider the demand correspondence $\mathbf{x}(\mathbf{p}, m)$ from Example 5.3.2. Determine the price range where good 2 is inferior. Show also that the Law of Demand fails there by calculating $\partial x_2/\partial p_2$.

Answer: The demand correspondence is

$$\mathbf{x}(\mathbf{p}, m) = \begin{cases} \left(\frac{m}{p_1}, 0\right) & \text{if } p_2 > p_1 \text{ or } m > 5p_1 \\ X(\mathbf{p}, m) & \text{if } p_1 = p_2 \text{ and } 5p_1 \leq m \\ \left(\frac{2m-5p_2}{2p_1-p_2}, \frac{5p_1-m}{2p_1-p_2}\right) & \text{if } p_2 < p_1 \text{ and } 5p_2/2 \leq m \leq 5p_1 \\ \left(0, \frac{m}{p_2}\right) & \text{if } p_2 < p_1 \text{ and } m \leq 5p_2/2 \end{cases}$$

where

$$X(\mathbf{p}, m) = \left\{\left(\alpha, \frac{m - \alpha p_1}{p_1}\right) \in \mathbb{R}^2_+ : 2m - 5 \leq \alpha p_1\right\}.$$

From the formula, we see that good two is inferior ($\partial x_2/\partial m < 0$) when $p_2 < p_1$ and $5p_2/2 \leq m \leq 5p_1$. The Law of Demand also fails there because $\partial x_2/\partial p_2 = m/(2p_1 - p_2)^2 > 0$.

5.3.4 Consider the utility function $u(x_1, x_2) = x_1 + \ln x_2$.

a) Find the Marshallian demand functions.

b) Use the Marshallian demands to compute the Slutsky matrix.

c) Show that the Slutsky matrix is negative semi-definite, but not negative definite.

Answer:

a) The Lagrangian is $\mathcal{L} = x_1 + \ln x_2 + \lambda(m - p_1x_1 - p_2x_2) + \mu_1x_1 + \mu_2x_2$ yielding first-order conditions $1 + \mu_1 = \lambda p_1$ and $1/x_2 + \mu_2 = \lambda p_2$. The second equation does not make sense if $x_2 = 0$, so we must have $x_2 > 0$. Complementary slackness then yields $\mu_2 = 0$. Substituting back in the first-order conditions, we find $\lambda = 1/(p_2x_2) > 0$, which implies $p_1x_1 + p_2x_2 = m$ by complementary slackness.

There are now two cases, $x_1 = 0$ and $x_1 > 0$. In the first case, the Marshallian demand is $\mathbf{x}(\mathbf{p}, m) = (0, m/p_2)$. The first-order conditions give $\lambda = 1/m$ and require $p_1 \le m$ in order that $\mu_1 \ge 0$. In the second case, $\lambda = 1/p_1$ yielding $x_2 = p_1/p_2$ and $x_1 = (m - p_1)/p_1$. Since $x_1 > 0$, this solution is only available if $m > p_1$.

Summing up, the Marshallian demand is:

$$\mathbf{x}(\mathbf{p}, m) = \begin{cases} (0, m/p_2) & \text{if } p_1 \ge m \\ \big((m - p_1)/p_1, p_1/p_2\big) & \text{if } p_1 < m. \end{cases}$$

b) We use the Slutsky equation $\partial h_k/\partial p_\ell = \partial x_k/\partial p_\ell + x_\ell \partial x_k/\partial m$. There are two cases: $p_1 \ge m$ and $p_1 < m$.

In the first case, the Slutsky matrix is zero. The only term where there is anything to compute is $\partial h_2/\partial p_2 = \partial x_2/\partial p_2 + x_2\partial x_2/\partial m = -m/p_2^2 + (m/p_2)(1/p_2) = 0$.

The second case is more interesting. Here

$$\mathbf{S} = \begin{bmatrix} -\frac{1}{p_1} & \frac{1}{p_2} \\ \frac{1}{p_2} & -\frac{p_1}{p_2^2} \end{bmatrix}.$$

c) When $p_1 \ge m$, S is the zero matrix, which is trivially negative semi-definite. Of course it is not definite as $S\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$.

When $p_1 < m$, both diagonal terms are negative and the determinant is zero. This sign pattern, which encompasses all of the principal minors, implies the matrix is negative semi-definite. The matrix is not definite as $\mathbf{S}(p_2, -p_1)^{\mathbf{T}} = \mathbf{0}$.

6.1.4 Let $A = \{(x, y) \in \mathbb{R}^2_+ : x^2 + y^2 \le 25\}$ and $\mathbf{x}_0 = (3, 4)$.

a) Show that A is a convex set.

b) Show that $\mathbf{x}_0 \notin \operatorname{int} A$.

c) Use Support Property II from section 27.6.1 to find a vector $\mathbf{p}$ as in Separation Theorem C.

Answer:

a) Now A is a lower contour set for the convex function $x^2 + y^2$, so it is a convex set.

b) $\operatorname{int} A = \{(x, y) : x^2 + y^2 < 25\}$. Since $3^2 + 4^2 = 25$, $\mathbf{x}_0 \notin \operatorname{int} A$.

c) Let $f(x, y) = -x^2 - y^2$ so $A = \{(x, y) : f(x, y) \geq f(\mathbf{x}_0) = -25\}$. Then the Support Property Theorem II tells us that $df(\mathbf{x}_0)\cdot\mathbf{x} \geq df(\mathbf{x}_0)\cdot\mathbf{x}_0$ for all $\mathbf{x} \in A$. Here $df = (-2x, -2y)$, so $df(\mathbf{x}_0) = (-6, -8)$ and $df(\mathbf{x}_0)\cdot\mathbf{x}_0 = -18 - 32 = -50$, so $(-6, -8)\cdot\mathbf{x} \geq -50$ for all $\mathbf{x} \in A$.