Homework #4

6.2.1 Let $g(\mathbf{x}) = f(\mathbf{x}-\mathbf{a})$ for some $\mathbf{a} \in \mathbb{R}^L$. Show that $g^*(\mathbf{p}) = \mathbf{p}\cdot\mathbf{a} + f^*(\mathbf{p})$.

Answer: Here

$$\begin{aligned} g^*(\mathbf{p}) &= \inf\{\mathbf{p}\cdot\mathbf{x} - g(\mathbf{x})\} \\ &= \inf_{\mathbf{x}}\{\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x}-\mathbf{a})\} \\ &= \inf_{\mathbf{x}}\{\mathbf{p}\cdot(\mathbf{x}+\mathbf{a}) - f(\mathbf{x})\} \\ &= \mathbf{p}\cdot\mathbf{a} + \inf_{\mathbf{x}}\{\mathbf{p}\cdot\mathbf{x} - f(\mathbf{x})\} \\ &= \mathbf{p}\cdot\mathbf{a} + f^*(\mathbf{p}). \end{aligned}$$

6.2.3 Let $A = \mathbb{R}^L_+$. Compute $\mathbb{I}^*_A(\mathbf{p})$.

Answer: Recall that

$$\mathbb{I}_A(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in A \\ -\infty & \text{if } \mathbf{x} \notin A. \end{cases}$$

Consider the problem of minimizing $\mathbf{p}\cdot\mathbf{x} - \mathbb{I}_A(\mathbf{x})$ by choice of $\mathbf{x}$. If $\mathbf{x} \notin A$, the minimum is $+\infty$! If $\mathbf{x} \in A$, we must minimize $\mathbf{p}\cdot\mathbf{x}$. Thus $\mathbb{I}^*_A(\mathbf{p}) = \inf\{\mathbf{p}\cdot\mathbf{x} : \mathbf{x} \in A\}$.

Now $A = \mathbb{R}^L_+$. For $\mathbf{p} \geq \mathbf{0}$, $\mathbf{p}\cdot\mathbf{x} \geq 0$ and $\mathbf{p}\cdot\mathbf{0} = 0$, so the minimum value is 0. If any $p_\ell < 0$, consideration of $\mathbf{p}\cdot(n\mathbf{e}_\ell) = np_\ell \to -\infty$ as $n \to \infty$ shows that the infimum is $-\infty$. It follows that

$$\mathbb{I}^*_A(\mathbf{p}) = \begin{cases} 0 & \text{when } \mathbf{p} \geq \mathbf{0} \\ -\infty & \text{otherwise.} \end{cases}$$

6.5.2 Let $e(\mathbf{p}, \bar{u}) = \bar{u}(p_1 + 3p_2)$. Find the conjugate function $e^*_{\bar{u}}(\mathbf{x})$.

Answer: We must minimize $\mathbf{p}\cdot\mathbf{x} - e(\mathbf{p}, \bar{u}) = p_1(x_1 - \bar{u}) + p_2(x_2 - 3\bar{u})$. Thus

$$e^*_{\bar{u}}(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1 \geq \bar{u} \text{ and } x_2 \geq 3\bar{u} \\ -\infty & \text{otherwise.} \end{cases}$$

Although not part of the problem, it is easy to see that the associated quasiconcave utility function is $u(\mathbf{x}) = \min\{x_1, x_2/3\}$.

8.1.3 Suppose expenditure has the form $e(\mathbf{p}, \bar{u}) = \bar{u}^\gamma a(\mathbf{p})$ where a is homogeneous of degree one in $\mathbf{p}$.

a) Is the utility function homogeneous? If so, what is the degree of homogeneity?

b) Using the results from part (1), compare the demand functions that correspond to the two expenditure functions $e_1(\mathbf{p}, \bar{u}) = \bar{u}a(\mathbf{p})$ and $e_2(\mathbf{p}, \bar{u}) = \bar{u}^2 a(\mathbf{p})$.

Answer:

a) We can find the utility function by taking the conjugate of $e_{\bar{u}}(\mathbf{p})$. The indicator function for the upper contour set at utility level $\bar{u}$ is

$$\begin{aligned} e^*_{\bar{u}}(\alpha\mathbf{x}) &= \inf_{\mathbf{p}}\{\mathbf{p}\cdot(\alpha\mathbf{x}) - \bar{u}^\gamma a(\mathbf{p})\} \\ &= \alpha \inf_{\mathbf{p}}\{\mathbf{p}\cdot\mathbf{x} - (\bar{u}/\alpha^{1/\gamma})^\gamma a(\mathbf{p})\} \\ &= \inf_{\mathbf{p}}\{\mathbf{p}\cdot\mathbf{x} - (\bar{u}/\alpha^{1/\gamma})^\gamma a(\mathbf{p})\} \\ &= e^*_{\bar{u}/\alpha^{1/\gamma}}(\mathbf{x}). \end{aligned}$$

Where line 3 follows due to the fact that e^* takes only the values 0 and $-\infty$. From this we conclude that $u(\alpha\mathbf{x}) \geq \bar{u}$ if and only if $\alpha^{1/\gamma}u(\mathbf{x}) \geq \bar{u}$. This implies u is homogeneous of degree $1/\gamma$ in $\mathbf{x}$.

b) In this case the utility function corresponding to e_1 is homogeneous of degree one while the function corresponding to e_2 is homogeneous of degree 2. They have the same $\bar{u} = 1$ level sets, so the second is just the square of the first. Preferences are identical and so is demand.