Homework #5

9.2.4 Suppose the production function can be written $f(\mathbf{z}) = F\big(\sum_\ell \phi_\ell(z_\ell)\big)$. Show that the marginal rate of technical substitution between any two inputs is independent of the amount of any other input used.

Answer: The marginal rate of technical substitution between k and ℓ is

$$\text{MRTS}_{k\ell} = \frac{\partial F/\partial z_k}{\partial F/\partial z_\ell} = \frac{F'(\sum_\ell \phi_\ell)\,\phi'_k(z_k)}{F'(\sum_\ell \phi_\ell)\,\phi'_\ell(z_\ell)} = \frac{\phi'_k(z_k)}{\phi'_\ell(z_\ell)},$$

which depends only on z_k and z_ℓ.

9.2.5 Consider the cost minimization problem for $f(z_1, z_2) = \sqrt{z_1 z_2}$ and $\mathbf{w} = (1, 0)$. Show that the cost minimization problem does not have a solution for any $q > 0$.

Answer: For $n > 0$, the input $\mathbf{z} = (q/n, qn)$ yields output q and its cost is $q/n \to 0$ as $n \to \infty$. The infimum of cost is zero, but this is never realized for any $\mathbf{z}$ with $\sqrt{z_1 z_2} \geq q$.

10.3.1 Suppose there are two inputs and one output with the Leontief production function $f(z_1, z_2) = \min(z_1, 3z_2)$. The output price is $p > 0$ and the input prices are $w_\ell > 0$.

a) Find all profit-maximizing net output vectors.

b) Calculate the profit function.

c) Show directly that the Law of Supply holds.

Answer: We will follow the convention of Example 11.1.1 and write $\mathbf{y} = (-z_1, -z_2, q)$ where q is the output and z_i are inputs. We write price as (w_1, w_2, p) and net output as $\mathbf{y} = (-z_1, -z_2, q)$.

a) Since both inputs are costly, cost is minimized when there is no excess of either input. Thus $z_1 = 3z_2 = q$ and cost is $w_1 q + w_2(q/3)$. Profit is then $(w_1, w_2, p)\cdot(-q, -q/3, q) = (p - w_1 - w_2/3)q$. There is no maximum when $p > w_1 + w_2/3$, and profit is maximized at $q = 0$ when $p < w_1 + w_2/3$. The net output correspondence is:

$$\mathbf{y}(\mathbf{p}) = \begin{cases} \{(-q, -q/3, q) : q \geq 0\} & \text{when } p = w_1 + w_2/3 \\ \{\mathbf{0}\} & \text{when } p < w_1 + w_2/3 \\ \text{undefined} & \text{when } p > w_1 + w_2/3. \end{cases}$$

b) Using part (a), we find the profit function is

$$\pi(p, \mathbf{w}) = \begin{cases} 0 & \text{for } p \leq w_1 + w_2/3 \\ +\infty & \text{otherwise.} \end{cases}$$

c) For $p < w_1 + w_2/3$, the net supply vector $\mathbf{y}$ is zero. Then $\Delta\mathbf{p}{\cdot}\Delta\mathbf{y} = \mathbf{p}'{\cdot}\mathbf{y}' - \mathbf{p}{\cdot}\mathbf{y}'$. Now $\mathbf{p}'{\cdot}\mathbf{y}' = 0$ due to constant returns to scale, and $\mathbf{p}{\cdot}\mathbf{y}' \le \mathbf{p}{\cdot}\mathbf{y} = 0$. It follows that $\Delta\mathbf{p}{\cdot}\Delta\mathbf{y} \ge 0$.

Similarly, if $p' < w_1' + w_2'/3$, $\mathbf{y}' = \mathbf{0}$ and $\Delta\mathbf{p}{\cdot}\Delta\mathbf{y} = -\mathbf{p}'{\cdot}\mathbf{y} \ge 0$.

Now suppose $p = w_1 + w_2/3$ and $p' = w_1' + w_2'/3$. Then $\Delta\mathbf{p} = (w_1' - w_1, w_2' - w_2, p' - p)$. Net outputs are $\mathbf{y} = (-q, -q/3, q)$ and $\mathbf{y}' = (-q', -q'/3, q')$, so $\Delta\mathbf{y} = \Delta q(-1, -1/3, 1)$. Now $\Delta\mathbf{p}{\cdot}\Delta\mathbf{y} = (\Delta p - \Delta w_1 - \Delta w_2/3)\Delta q$. Then $\Delta p - \Delta w_1 - \Delta w_2/3 = 0$, so $\Delta\mathbf{p}{\cdot}\Delta\mathbf{y} = 0$.

Either way, the Law of Supply holds.

11.1.1 Suppose production is described by a linear activity model with three activities: $\mathbf{a}^1 = (3, 0, -1)^{\mathbf{T}}$, $\mathbf{a}^2 = (0, 2, -1)^{\mathbf{T}}$ and $\mathbf{a}^3 = (1, 1, -1/2)^{\mathbf{T}}$.

a) Find the production set's dual cone and find a non-trivial price vector in the dual cone.

b) At what prices in the dual cone may activity 1 be utilized?

c) At what prices in the dual cone may activity 2 be utilized?

d) At what prices in the dual cone may activity 3 be utilized?

e) At what prices in the dual cone may both activities 1 and 2 be utilized?

f) Are there any non-zero prices in the dual cone where all activities may be utilized?

Answer:

a) We need to find price vectors with $\mathbf{p}{\cdot}\mathbf{a}^i \le 0$ for $i = 1, 2, 3$. This requires

$$\begin{aligned} 3p_1 - p_3 &\le 0 \\ 2p_2 - p_3 &\le 0 \\ p_1 + p_2 - p_3/2 &\le 0. \end{aligned}$$

A non-trivial price vector in the dual cone is $\mathbf{p} = (1/4, 1/3, 2)^{\mathbf{T}}$. Note that $\mathbf{p}{\cdot}\mathbf{a}^i \le 0$ for all i.

b) For the next three parts, we need to satisfy the zero profit condition for the activity in question. In the case of activity 1, this requires $3p_1 = p_3$.

c) This requires $2p_2 = p_3$.

d) This requires $p_1 + p_2 = p_3/2$.

e) Only price vectors proportional to $(2, 3, 6)^{\mathbf{T}}$ satisfy this. They are the only vectors in the dual cone that obey the conditions in both parts (b) and (c).

f) No. The vectors in part (e) fail to satisfy part (d).