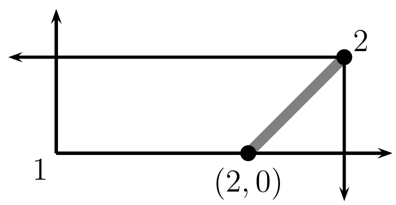
# Homework #7

15.1.2 Suppose there are three goods and utility has the Cobb-Douglas forms $u_1(\mathbf{x}^1) = (x_1^1)^{1/3}(x_2^1)^{1/3}(x_3^1)^{1/3}$, $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{1/3}(x_3^2)^{1/3}$, and $\boldsymbol{\omega} = (5, 2, 3)$. Find all Pareto optimal allocations.

**Answer:** We appeal to Example 16.1.5. Each consumer receives a share of the endowment. The Pareto set is $\{(\mathbf{x}^1, \mathbf{x}^2) : \mathbf{x}^1 = t(5, 2, 3), \mathbf{x}^2 = (1 - t)(5, 2, 3), 0 \leq t \leq 1\}$.

15.1.6 Suppose $u_1(\mathbf{x}^1) = x_1^1 + 2x_2^1$ and $u_2(\mathbf{x}^2) = \min\{x_1^2, x_2^2\}$, with $\boldsymbol{\omega} = (3, 1)$. Find all Pareto optimal allocations.

**Answer:** $\text{MRS}_{12}^1 = 1/2$ while $\text{MRS}_{12}^2$ can be interpreted as anything when $x_1^2 = x_2^2$. The interior Pareto optimal allocations run from the upper right corner of the box to $(2, 0)$. The two boundary points are included. Note that $(x, 0)$ for $x < 2$ is not Pareto optimal as $(2, 0)$ is a Pareto improvement (consumer 1 is better off, consumer 2 is indifferent). The set of Pareto optima is the heavy line in the diagram, $\{(x_1, x_2) : x_1 - 2 = x_2, x_1 \geq 2\}$.




16.1.2 An economy has two goods and two identical Cobb-Douglas consumers with $u_i(\mathbf{x}^i) = \sqrt{x_1^i x_2^i}$. The total endowment is $(0, 6)$. There is one constant returns to scale firm that produces good 1 and uses good 2 as its only input. The production function is $f(z) = 5z$.

*a*) Find all Pareto optimal allocations of goods and the corresponding net output vector.

*b*) Find prices and wealth levels that make each Pareto optima a quasi-equilibrium with taxes and transfers.

**Answer:**

*a*) The net output vector will have the form $\mathbf{y} = (-5y_2, y_2)$ with $y_2 \leq 0$. Here the marginal rate of transformation is $\text{MRT}_{12} = 1/5$. This must also be the marginal rate of substitution at any interior Pareto optimum. Note that since utility is zero if there is no production, the production technology must be used.

Now $\text{MRS}_{12}^i = x_2^i/x_1^i = 1/5$, so $5x_2^i = x_1^i$. Summing over both consumers, aggregate consumption obeys $5x_2 = x_1$. Since good 1 can only be obtained from the production sector, $x_1 = -5y_2$ and $x_2 = 6 + y_2$. Thus $x_2 = -y_2$ and $x_2 = 6 + y_2$. It follows that $x_2 = 3$ and $y_2 = -3$, so $x_1 = 15$.

Since both consumers will consume in the same proportions, $\mathbf{x}_1 = \alpha(15, 3)$ and $\mathbf{x}_2 = (1-\alpha)(15, 3)$ for some $\alpha$ between 0 and 1. Also, $\mathbf{y} = (15, -3)$. These are the Pareto optimal allocations. The cases $\alpha = 0$ and $\alpha = 1$ are the Pareto optima on the border of the Edgeworth box.

*b*) We have already derived implicitly the prices in part (a), $\mathbf{p} = (1, 5)$. The wealth levels that yield the goods allocation $\mathbf{x}^1 = \alpha(15, 3)$ and $\mathbf{x}^2 = (1-\alpha)(15, 3)$ are $m^1 = 30\alpha$ and $m^2 = 30(1-\alpha)$.

17.1.1 Suppose $u_1(\mathbf{x}^1) = (x_1^1)^{1/3}(x_2^1)^{2/3}$ and $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$, with endowments $\boldsymbol{\omega}^1 = (7, 1)$ and $\boldsymbol{\omega}^2 = (3, 1)$. Find the core.

**Answer:** Since the consumers have identical Cobb-Douglas preferences, the Pareto set is the diagonal of the Edgeworth box. The aggregate endowment is $\boldsymbol{\omega} = (10, 2)$ and total utility is $10^{1/3}2^{2/3} = 40^{1/3}$. Individual rationality requires $u_1 \geq u_1(\boldsymbol{\omega}^1) = 7^{1/3}$ and $u_2 \geq u_2(\boldsymbol{\omega}^2) = 3^{1/3}$. Thus the core is $\mathfrak{C}(\mathcal{E}) = \{\big(u_1(10,2), (1 - u_1(10,2))\big) : u_1 \geq 7^{1/3}, u_2 \geq 3^{1/3}, u_1 + u_2 = 40^{1/3}\}$