Homework #1

3.1.4 Let $f(x_1, x_2) = x_1/(x_1 + x_2) + \ln(x_1 + x_2)$ on $\mathbb{R}^2_{++}$. Show that f is increasing and has derivative that is homogeneous of degree minus one.

Answer: The derivative is

$$\begin{aligned} df &= \left(\frac{x_2}{(x_1+x_2)^2}, -\frac{x_1}{(x_1+x_2)^2}\right) + \left(\frac{1}{x_1+x_2}, \frac{1}{x_1+x_2}\right) \\ &= \frac{1}{(x_1+x_2)^2}\big(x_1 + 2x_2, x_2\big) \gg \mathbf{0} \end{aligned}$$

on $\mathbb{R}^2_{++}$. Now

$$\begin{aligned} df(t\mathbf{x}) &= \frac{1}{t^2(x_1+x_2)^2}\big(tx_1 + 2tx_2, tx_2\big) \\ &= t^{-1}\frac{1}{(x_1+x_2)^2}\big(x_1 + 2x_2, x_2\big) \\ &= t^{-1} df(\mathbf{x}), \end{aligned}$$

so df is homogeneous of degree minus one.

3.2.2 Suppose u is additive separable on $\mathbb{R}^L_+$.

a) True or False. For each good ℓ, marginal utility $\text{MU}_\ell = \partial u/\partial x_\ell$ is diminishing in x_ℓ.

b) If you answered True to (a), prove it. If you answered False, provide a counter-example.

Answer:

a) **False**.

b) Consider $u(\mathbf{x}) = \sum_{\ell=1}^L x_\ell^2$. This is additive because $\partial^2 u/\partial x_k \partial x_\ell = 0$ for all $k \neq \ell$. The marginal utility of good ℓ is $\text{MU}_\ell = 2x_\ell$, which is increasing in x_ℓ.

Even if we require we also require convexity of u, so that $\partial^2 u/\partial x_k \partial x_\ell \leq 0$, there is still the borderline case $u(\mathbf{x}) = \sum_{\ell=1}^L \alpha_\ell x_\ell$ where $\text{MU}_\ell = \alpha_\ell$ is constant.

3.3.2 Suppose $u(\mathbf{x}) = x_1 + x_2(x_3 + x_4)$ on $\mathbb{R}^L_{++}$.

a) Find all commodity groups A in $\{1, 2, 3, 4\}$ where u induces an order on A.

b) For each commodity group A in part (a), find a subutility function that represents $\succsim_A$.

c) Rewrite u in terms of the non-trivial subutilities you found in (b). If there is more than one way to do this, demonstrate all of them.

Answer:

a) For simplicity, denote $\partial u/\partial x_k$ by u_k. The marginal utilities are $u_1 = 1$, $u_2 = x_3 + x_4$, $u_3 = x_2$ and $u_4 = x_2$. Since each marginal utility is strictly positive, u is strictly increasing and induces an order on each singleton.

We also find the following marginal rates of substitution involving x_1: $u_1/u_2 = 1/(x_3 + x_4)$, $u_1/u_3 = 1/x_2$, $u_1/u_4 = 1/x_2$. This means that any subutility that includes x_1 as an argument must include all other goods.

For $A = \{2, 3, 4\}$, we also have $\text{MRS}_{23} = \text{MRS}_{24} = (x_3 + x_4)/2$ and $\text{MRS}_{34} = 1$. These are all independent of x_1, meaning that u induces an order on $\{2, 3, 4\}$. Moreover, any group including good 2, must include both goods 3 and 4.

Finally, for $A = \{3, 4\}$, $u_3/u_4 = 1$, which is independent of x_1 and x_2, so u induces an order on $\{3, 4\}$.

b) For the singleton groups, the trivial subutility $v_k(x_k) = x_k$ will do. For $A = \{3, 4\}$, the subutility $\phi(x_3, x_4) = x_3 + x_4$ works. For $B = \{2, 3, 4\}$, the subutility $\psi(x_2, x_3, x_4) = x_2(x_3 + x_4)$ will do.

c) We can write $u(\mathbf{x}) = x_1 + \psi(x_2, x_3, x_4) = x_1 + x_2\phi(x_3, x_4)$. That is $u = x_1 + \psi$ or $u = x_1 + x_2\phi$.

3.3.3 Suppose utility on $\mathbb{R}^3_+$ is given by $u(\mathbf{x}) = (x_1 + 1)x_2(x_3 + 5)$.

a) Is there a monotonic transformation that transforms u into an additive separable utility function?

Answer: Yes. Let $v = \ln u$. Then $v(\mathbf{x}) = \ln(x_1 + 1) + \ln x_2 + \ln(x_3 + 5)$, which is in additive separable form.

If you can't quickly guess it, one way to find the right transformation is to consider $v(\mathbf{x}) = \phi\big(u(\mathbf{x})\big)$. The second cross partial derivatives of v must be zero. Now $\partial v/\partial x_1 = \phi' x_2(x_3 + 5)$ and $\partial^2 v/\partial x_2 \partial x_1 = \phi'(x_3 + 5) + \phi''(x_1 + 1)x_2(x_3 + 5)^2 = 0$. This can be written as $\phi'(x_3 + 5) + \phi'' u(x_3 + 5) = 0$. Thus $\phi' + \phi'' u = 0$. Let $\psi = \phi'$ so that $\psi + \psi' u = 0$. In other words, $d\psi/\psi = -du/u$. The solution is $\psi(u) = A/u$ for some constant A which may be of either sign. Now $\phi' = \psi = A/u$. This has general solution $\phi = B + A \ln u$ for some constants A and B. Because ϕ is increasing, $A > 0$. One such function is $\phi(u) = \ln u$. We don't have to worry about the other cross partial derivatives as ϕ converts u into the additive separable form $v(\mathbf{x}) = \ln(x_1 + 1) + \ln x_2 + \ln(x_3 + 5)$.

b) Is u completely separable?

Answer: Yes. Since it is additive separable it is also completely separable.

To answer more fully, it induces the same preference order defined by the utility function x_i on $\{i\}$. On $\{1, 2\}$, it induces the order defined by the utility function $\ln(x_1 + 1) + \ln x_2$. On $\{1, 3\}$ it induces $\ln(x_1 + 1) + \ln(x_3 + 5)$. On $\{2, 3\}$ it induces $\ln x_2 + \ln(x_3 + 5)$. Since it induces an order on every commodity subgroup of $\{1, 2, 3\}$ it is completely separable.