Homework #10

25.1.2 Consider a pure exchange economy with two consumers, one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 2)$. Both consumers have identical utility function $u(\mathbf{x}) = \pi \ln x_1 + (1 - \pi) \ln x_2$ where $0 < \pi < 1$ is the probability of state one.

a) Find all Arrow-Debreu equilibria.

b) How does the equilibrium price of good two relative to good one relate to the probability π?

Answer:

a) Let $\mathbf{p} \gg \mathbf{0}$ be the price vector. Note that zero price is not allowed in equilibrium due to the Cobb-Douglas preferences. Since preferences are Cobb-Douglas and identical for both consumers, so demand is $\mathbf{x}^i = m^i\big(\pi/p_1, (1 - \pi)/p_2\big)$ where $m^i = \mathbf{p}\cdot\boldsymbol{\omega}^i$ is the income of consumer i. It follows that market demand is $m\big(\pi/p_1, (1 - \pi)/p_2\big)$ where $m = m^1 + m^2$.

Market supply is $\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2 = (2, 2)$. Setting demand equal to supply we find $\pi m/p_1 = (1 - \pi)m/p_2 = 2$. Using good one as numéraire, $m = 2/\pi$, the equilibrium prices are $\mathbf{p} = \big(1, (1 - \pi)/\pi\big)$. Individual incomes are $m^1 = 2$ and $m^2 = 2(1 - \pi)/\pi$ and the corresponding allocation is $\mathbf{x}^1 = \big(2\pi, 2\pi\big)$ and $\mathbf{x}^2 = \big(2(1 - \pi), 2(1 - \pi)\big)$.

Any positive scalar multiple of $\mathbf{p}$ is also an equilibrium price vector with the same allocation.

b) From part (a), the relative price of good two is $p_2/p_1 = (1 - \pi)/\pi$.

25.1.4 Consider a pure exchange economy with one good and two states. Endowments are $\boldsymbol{\omega}^1 = (2, 0)$ and $\boldsymbol{\omega}^2 = (0, 5)$. Both consumers have utility $u(\mathbf{x}) = \ln x_1 + \ln x_2$.

a) Find the Arrow-Debreu equilibrium.

b) Are the consumers fully insured?

Answer:

a) With Cobb-Douglas utility, both prices must be strictly positive. We take good one as numéraire and set $\mathbf{p} = (1, p)$. Then demand is

$$\mathbf{x}^1 = (1, 1/p) \quad \text{and} \quad \mathbf{x}^2 = \frac{5p}{2}(1, 1/p).$$

Setting supply equal to demand for good one we obtain $1 + 5p/2 = 2$, so $p = 2/5$. Equilibrium prices are $\hat{\mathbf{p}} = (1, 2/5)$. Then $\hat{\mathbf{x}}^1 = (1, 5/2)$ and $\hat{\mathbf{x}}^2 = (1, 5/2)$.

b) Here there is aggregate uncertainty and full insurance for both consumers is not possible. In fact, neither consumers is fully insured since $\hat{x}_1^i \neq \hat{x}_2^i$.

25.2.7 Consider an exchange economy with two consumers, one good, and 3 states of the world. Let x_s^i denote consumer i's consumption of the one good in state s. Each consumer has utility function

$$u(\mathbf{x}^i) = \sum_{s=1}^{3} \ln x_s^i.$$

The endowments are $\boldsymbol{\omega}^1 = (1, 2, 3)$ and $\boldsymbol{\omega}^2 = (3, 2, 1)$. Find the Arrovian securities equilibrium. Be sure to indicate the spot market prices, prices of the securities, and the equilibrium consumption by each consumer.

Answer: With $p_s = 1$ for every s, both consumers consume their income in state s. I.e., $x_{1,s}^i = \boldsymbol{\omega}_{1,s}^i + z_s^i$. Thus indirect utility is $v^i(\mathbf{z}) = \sum_{s=1}^3 \ln(\boldsymbol{\omega}_{1,s}^i + z_s^i)$. The first-order conditions are then $\lambda_i q_s = 1/(\boldsymbol{\omega}_{1,s}^i + z_s^i)$ or $q_s \boldsymbol{\omega}_{1,s}^i + q_s z_s^i = 1/\lambda_i$. Summing over i and using asset market clearing, we find $4q_s = 1/\lambda_1 + 1/\lambda_2$. The q_s's must be equal, so we may set $q_s = 1$ for every security s.

Now $1/\lambda_i = \omega_{1,s}^i + z_s^i = \omega_{1,s}^i + q_s z_s^i$. Summing over s and using the asset budget constraint we find $3/\lambda_i = 6$. Thus $1/\lambda_i = 2$, so $2 = \boldsymbol{\omega}_{1,s}^i + z_s^i$. It follows that $\mathbf{z}^1 = (1, 0, -1)$ and $\mathbf{z}^2 = (-1, 0, 1)$. The corresponding goods allocations are $\mathbf{x}^1 = (2, 2, 2)$ and $\mathbf{x}^2 = (2, 2, 2)$.

26.1.2 Consider a pure exchange economy with one good and three states. Endowments are $\boldsymbol{\omega}^1 = (2, 2, 0)$ and $\boldsymbol{\omega}^2 = (1, 1, 2)$. Consumer one has utility $u_1(\mathbf{x}) = \ln x_1 + \ln x_2 + \ln x_3$ and consumer two has utility $u_2(\mathbf{x}) = \min\{x_1, x_2, x_3\}$. In other words, consumer two is infinitely risk averse The return matrix is $\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

a) Find the Radner equilibrium $(\hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\mathbf{x}}^i, \hat{\mathbf{z}}^i)$.

b) Are the consumers fully insured?

c) Is the equilibrium allocation Pareto optimal?

Answer:

a) The price of the one good cannot be zero in any spot market as that would lead to infinite demand by consumer one. Since the price is positive, we use good one as numéraire in every spot market. The equilibrium spot prices are then $\hat{\mathbf{p}} = (1, 1, 1)$.

We will denote $x^i_{1,s}$ by x^i_s. The return matrix yields the following budget constraints:

$$x^i_1 \le \omega^i_1 + z^i_1$$
$$x^i_2 \le \omega^i_2$$
$$x^i_1 \le \omega^i_3 + z^i_2.$$

Utility is maximized when the entire budget is spent. This yields indirect utility

$$v_1(\mathbf{z}^1) = \ln(2 + z^1_1) + \ln 2 + \ln z^1_2$$
$$v_2(\mathbf{z}^2) = \min\{(1 + z^2_1), 1, (2 + z^2_2)\}.$$

It is clear that consumer one will demand an infinite amount of asset two unless its price is positive. This allows us to use asset two as numéraire in the asset market, so we set $\mathbf{q} = (q, 1)$. Then the asset budget constraint becomes $z^i_1 = -qz^i_2$.

Maximizing v_1 with $z^1_1 = -qz^1_2$, we find $\mathbf{z}^1 = (-1, 1/q)$. Now $v_2(z^2_2) = \min\{(1 - qz^2_2), 1, (2 + z^2_2)\}$, which is maximized by any z^2_2 with $-1 \le z^2_2 \le 0$. Market demand for asset one is $-1 - qz^2_2$, which must be zero by market clearing. Thus $z^2_2 = -1/q$ and we must have $q \ge 1$ since $-1 \le z^2_2$.

As a result, any $\hat{\mathbf{q}} = (1, q)$ with $q \ge 1$ is an equilibrium asset price vector, with asset demands $\hat{\mathbf{z}}^1 = (-1, 1/q)$ and $\hat{\mathbf{z}}^2 = (+1, -1/q)$, which clears the asset market. The corresponding allocations are $\hat{\mathbf{x}}^1 = (1, 2, 1/q)$ and $\hat{\mathbf{x}}^2 = (2, 1, 2 - 1/q)$.

b) Neither consumer is fully insured. Aggregate consumption must be different in different states, so some consumer must have consumption that differs across states.

c) The equilibrium allocation is not Pareto optimal when $q > 1$. In fact, the equilibrium with $q = 1$ Pareto dominates all equilibria with $q > 1$. The $q = 1$ equilibrium allocation is not Pareto optimal either! It is Pareto dominated by $\mathbf{x}^1 = (2, 2, 1)$, $\mathbf{x}^2 = (1, 1, 1)$.