Homework #2

5.2.2 Suppose indirect utility has the Gorman form $v(\mathbf{p}, m) = a(\mathbf{p}) + mb(\mathbf{p})$ for some functions $a(\mathbf{p})$ and $b(\mathbf{p})$.

a) Show that the expenditure function has a similar form, $e(\mathbf{p}, \bar{u}) = c(\mathbf{p}) + \bar{u}d(\mathbf{p})$ for some functions $c(\mathbf{p})$ and $d(\mathbf{p})$.

b) What degrees of homogeneity must c and d have for Theorem 5.1.5 to hold?

c) What does (b) imply about the homogeneity of a and b? Is this consistent with Theorem 4.5.1?

Answer:

a) Using duality, we may write $\bar{u} = v(\mathbf{p}, e(\mathbf{p}, \bar{u}) = a(\mathbf{p}) + mb(\mathbf{p})$. Solving for $e(\mathbf{p}, \bar{u})$, we find $e(\mathbf{p}, \bar{u}) = c(\mathbf{p}) + \bar{u}d(\mathbf{p})$ where $c(\mathbf{p}) = -a(\mathbf{p})/b(\mathbf{p})$ and $d(\mathbf{p}) = 1/b(\mathbf{p})$.

b) For e to be homogeneous of degree 1 in $\mathbf{p}$, both c and d must be homogeneous of degree 1 in $\mathbf{p}$ (set $\bar{u} = 0$ to see this for c, it then follows for d).

c) Then b must be homogeneous of degree -1 in $\mathbf{p}$ while a is homogeneous of degree 0. This implies v is homogeneous of degree 0 in $(\mathbf{p}, m)$, as required by the theorem.

5.2.3 Suppose $v(\mathbf{p}, m)$ is homogeneous of degree γ in m. Show that $e(\mathbf{p}, \bar{u})$ is homogeneous of degree $1/\gamma$ in $\bar{u}$. Is the converse true?

Answer: Yes, the converse is true. To show it, we use duality. We know $v\big(\mathbf{p}, e(\mathbf{p}, \bar{u})\big) = \bar{u}$. Now if e is homogeneous of degree $1/\gamma$ in $\bar{u}$, $t\bar{u} = v\big(\mathbf{p}, e(\mathbf{p}, t\bar{u})\big) = v\big(\mathbf{p}, t^{1/\gamma}e(\mathbf{p}, \bar{u})\big)$ which implies $tv\big(\mathbf{p}, e(\mathbf{p}, \bar{u})\big) = v\big(\mathbf{p}, t^{1/\gamma}e(\mathbf{p}, \bar{u})\big)$. This last equation tells us that v is homogeneous of degree γ in m.

9.1.2 Suppose consumers have indirect utility of the form $v_i(\mathbf{p}, m^i) = a_i(\mathbf{p})^2 + 2a_i(\mathbf{p})b(\mathbf{p})m + m^2b(\mathbf{p})^2$ for $m^i \in M_i$, an open interval, with the function a homogeneous of degree zero in $\mathbf{p}$ and the function b homogeneous of degree -1 in $\mathbf{p}$. Is strong aggregation possible here?

Answer: Yes. Note that $v_i(\mathbf{p}, m^i) = \big(a_i(\mathbf{p}) + mb(\mathbf{p})\big)^2$. This is equivalent to the indirect utility function $w_i(\mathbf{p}, m^i) = a_i(\mathbf{p}) + mb(\mathbf{p})$, which allows strong aggregation by Proposition 9.1.1.

A second way to obtain the result is to use Roy's Identity. Here

$$\frac{\partial v_i}{\partial p_\ell} = 2(a_i + mb)\left(\frac{\partial a_i}{\partial p_\ell} + m\frac{\partial b}{\partial p_\ell}\right)$$

and

$$\frac{\partial v_i}{\partial m} = 2(a_i + mb)\, b(\mathbf{p}).$$

Roy's Identity yields

$$x_\ell = -\frac{1}{b}\frac{\partial a_i}{\partial p_\ell} - m\frac{1}{b}\frac{\partial b}{\partial p_\ell},$$

which has the necessary form.

9.2.1 Suppose $\boldsymbol{\omega} = (1,0)$, $\mathbf{p} = (1,0)$, $\mathbf{p}' = (0,1)$ and utility has the Leontief form $u(\mathbf{x}) = \min\{x_1, 2x_2\}$.

a) Does the Law of Demand hold for Walrasian demand $\boldsymbol{\xi}(\mathbf{p}, \boldsymbol{\omega})$?

b) Does the Law of Demand hold when the prices are $\mathbf{p} = (1, p)$ and $\mathbf{p}' = (p, 1)$ for $0 < p < 1$?

Answer:

a) **No.** The Law of Demand does not hold here. Income in the two cases is $m = \mathbf{p}{\cdot}\boldsymbol{\omega} = 1$ and $m' = \mathbf{p}'{\cdot}\boldsymbol{\omega} = 0$. Thus demand obeys $\boldsymbol{\xi} = (1, 1/2) \in \boldsymbol{\xi}(\mathbf{p}, 1)$ and $\boldsymbol{\xi}' = (0,0) \in \boldsymbol{\xi}(\mathbf{p}', 0)$. But $(\mathbf{p}' - \mathbf{p}){\cdot}(\boldsymbol{\xi}' - \boldsymbol{\xi}) = (-1, +1){\cdot}(-1, -1/2) = 1/2 > 0$, violating the Law of Demand.

b) **No.** The Law of Demand still does not hold even though demand is now a function. Income is now $m = 1$ and $m' = p$ It follows that demand is now

$$\begin{aligned}
\xi &= \frac{1}{2+p}(2,1) \\
\xi' &= \frac{p}{2p+1}(2,1).
\end{aligned}$$

Now $\mathbf{p}' - \mathbf{p} = (p-1)(1,-1)$ and $\xi' - \xi = [(2+p)^{-1} - p(2p+1)^{-1}](2,1)$. It follows that

$$\begin{aligned}
(\mathbf{p}' - \mathbf{p}){\cdot}(\boldsymbol{\xi}' - \boldsymbol{\xi}) &= (p-1)\left(\frac{p}{2p+1} - \frac{1}{2+p}\right)\Big[(1,-1){\cdot}(2,1)\Big] \\
&= (p-1)\frac{p^2-1}{(2+p)(2p+1)} \\
&= \frac{(p+1)(p-1)^2}{(2+p)(2p+1)}.
\end{aligned}$$

This is positive when $p \neq 1$, showing that the Law of Demand is violated.