Homework #3

10.4.1 Suppose $F(x) = ax$ for $0 \le x \le 10$, $F(x) = 0$ for $x \le 0$ and $F(x) = 10a$ for $x \ge 10$.

a) Find the value of a that makes F a c.d.f.

b) Let $u(x) = x^2$. Using the value of a from part (a), compute $u(F)$.

Answer:

a) Since a c.d.f. must have limit 1 at $+\infty$, a must be 1/10.

b) Here

$$u(F) = \int_0^{10} \frac{1}{10} x^2 dx = \frac{1}{30} x^3 \Big|_0^{10} = \frac{1}{30}(10^3 - 0) = \frac{100}{3}.$$

10.4.3 Suppose a lottery has probability density $f(x) = 3x^2$ for $0 \le x \le 1$ and zero otherwise. Let $u(x) = x^{1/4}$. Compute the expected utility.

Answer: The expected utility is

$$\begin{aligned} Eu &= \int_0^1 u(x) f(x)\, dx = \int_0^1 x^{1/4}\,(3x^2)\, dF(x) \\ &= \int_0^1 3x^{9/4}\, dx = \left[\frac{12}{13} x^{13/4}\right]_0^1 \\ &= 12/13. \end{aligned}$$

11.1.2 Suppose F is uniformly distributed over $[1, 10]$. Calculate the expected utility $u(F)$ and the certainty equivalent $c(u, F)$ for the following utility functions.

a) $u(x) = 15x$.

b) $u(x) = \ln x$.

c) $u(x) = 20x - x^2$.

d) $u(x) = x^2$.

Answer: The distribution function is

$$F(x) = \begin{cases} 0 & \text{when } x \le 1 \\ (x-1)/9 & \text{when } 1 \le x \le 10 \\ 1 & \text{when } x \ge 10. \end{cases}$$

It follows that $dF = dx/9$ and that $EF = (1/9)\int_1^{10} x\, dx = (1/18)x^2\big|_1^{10} = 11/2$. We are now ready to tackle the four utility functions.

a) Here $Eu = (15/9)\int_1^{10} x\,dx = (5/6)x^2\Big|_1^{10} = 165/2$. The certainty equivalent is found by solving $165/2 = u(c) = 15c$, so $c = 11/2$. The risk neutrality of linear utility means the certainty equivalent is the expected value.

b) Here $Eu = (1/9)\int_1^{10} \ln x\,dx = (1/9)\big[x\ln x - x\big]_1^{10} = (10/9)\ln 10 - 1$. The certainty equivalent is found by solving $u(c) = \ln c = (10/9)\ln 10 - 1$. So $c = \exp[(10/9)\ln 10 - 1] = 10^{10/9}/e$. Here $c \approx 4.75 < 11/2$. This consumer is risk averse.

c) Here $Eu = (1/9)\int_1^{10} 20x - x^2\,dx = (1/9)\big[10x^2 - x^3/3\big]_1^{10} = \frac{1}{9}[1000 - 1000/3 - (10 - 1/3)] = \frac{1}{9}[2001/3 - 10] = 73$. The certainty equivalent is found by solving $u(c) = 20c - c^2 = 73$. The solutions are $10 \pm 3\sqrt{3}$. Oddly enough, this utility function has two certainty equivalents, one where utility is increasing, the other where utility is decreasing. You will recall that some of the theorems regarding various notions of risk aversion required $u' > 0$. This problem shows what can happen otherwise.

d) Here $Eu = (1/9)\int_1^{10} x^2\,dx = (1/9)\big[x^3/3\big]_1^{10} = \frac{1}{27}[1000 - 1] = \frac{1}{27}[999] = 111/3 = 37$. The certainty equivalent is found by solving $u(c) = c^2 = 37$ so $c = \sqrt{37}$.

11.1.4 Suppose F is uniformly distributed over $[1, a]$ for $a > 1$. Calculate the risk premium for the following utility functions.

a) $u(x) = x^3$.

b) $u(x) = x^{1/2}$.

c) $u(x) = \ln x$.

Answer: Note that the probability density for all three parts is $1/(a-1)$, and that this density has expected value of $(1+a)/2$.

a) The expected utility is $(\int_1^a x^3\,dx)/(a-1) = (a^4-1)/4(a-1)$. This has certainty equivalent $[(a^4-1)/4(a-1)]^{1/3}$, so the risk premium is $R(u,F) = (1+a)/2 - [(a^4 - 1)/4(a-1)]^{1/3}$.

b) The expected utility is $(\int_1^a x^{1/2}\,dx)/(a-1) = 2(a^{3/2}-1)/2(a-1)$. This has certainty equivalent $[2(a^{3/2}-1)/2(a-1)]^2$, so the risk premium is $R(u,F) = (1+a)/2 - [2(a^{3/2} - 1)/2(a-1)]^2$.

c) The expected utility is $(\int_1^a \ln x\,dx)/(a-1) = -1 + \ln a/(a-1)$. This has certainty equivalent a^{a-1}/e, so the risk premium is $R(u,F) = (1+a)/2 - a^{a-1}/e$.

11.2.2 Suppose a firm is uncertain about its costs. Specifically, suppose costs are $\alpha c(q)$ where α is a positive random variable and c is a $\mathcal{C}^2$ cost function obeying $c', c'' > 0$. Suppose the firm is risk neutral. Set up and solve the firm's problem. How does the solution with this type of uncertainty compare to the case where α is replaced by $E\alpha$, and thus known with certainty.

Answer: Suppose the firm is a price-taker with output price p. Expected profit from output q are $\int [pq - \alpha c(q)]\, dF(\alpha) = pq - (E\alpha)c(q)$. Since the firm is risk neutral, it maximizes expected profit (not the expected utility from profit). The first-order conditions are $p = (E\alpha)c'(q)$, which is precisely what a firm with certain cost of $(E\alpha)c(q)$ would do.