Homework #4

14.1.1 Show the following:

a) If Y is additive and divisible, then Y is a convex set.

b) If Y is convex and satisfies inaction, it also satisfies divisibility.

Answer:

a) Let $0 \le \alpha \le 1$ and $\mathbf{y}, \mathbf{y}' \in Y$. By divisibility, both $\alpha\mathbf{y} \in Y$ and $(1-\alpha)\mathbf{y}' \in Y$. We can then use additivity to see that the convex combination $\alpha\mathbf{y} + (1-\alpha)\mathbf{y}'$ is in Y. This shows that Y is convex.

b) For $0 \le \alpha \le 1$ and $\mathbf{y} \in Y$, we can write $\alpha\mathbf{y} = (1-\alpha)\mathbf{0} + \alpha\mathbf{y}$. Since $\mathbf{0} \in Y$, $\alpha\mathbf{y}$ is then a convex combination of two elements of Y, $\mathbf{0}$ and $\mathbf{y}$. As such, it is in Y, establishing divisibility.

14.3.2 Suppose there are two inputs and one output with the linear production function $f(z_1, z_2) = 2z_1 + z_2$. The output price is $p > 0$ and the input prices are $w_\ell > 0$.

a) Find all profit-maximizing net output vectors.

b) Calculate the profit function.

c) Show directly that the Law of Supply holds.

Answer:

a) Because this is a linear technology (CRS), maximum profit is either zero or infinite. To maximize profits, output must be $q = 2z_1 + z_2$ with profit $p(2z_1 + z_2) - w_1 z_1 - w_2 z_2$. Profits will be infinite if either $2p > w_1$ or $p > w_2$. We can also see that if both $2p < w_1$ and $p < w_2$, only $\mathbf{0}$ maximizes profit.

Finally, if $2p = w_1$, any amount of input 1 is optimal while if $p = w_2$, any amount of input 2 is optimal.

b) The profit function is

$$\pi(\mathbf{p}) = \begin{cases} +\infty & \text{if } 2p > w_1 \text{ or } p > w_2 \\ 0 & \text{if } 2p \le w_1 \text{ and } p \le w_2. \end{cases}$$

c) Profit maximization is only possible when $2p \le w_1$ and $p \le w_2$. We write $\Delta\mathbf{p}\cdot\Delta\mathbf{y} = [2(p'-p) - w_1' + w_1](z_1' - z_1) + [p' - p - w_2' + w_2](z_2' - z_2)$. Notice that the w_1 and w_2 terms act independently. We will consider the w_1 terms, but similar arguments apply to the w_2 terms.

There are 4 cases for the w_1 terms. (1) If $2p' = w_1'$ and $2p = w_1$, the term is zero. (2) If $2p' = w_1'$ and $2p < w_1$, $z_1 = 0$, so we have $(-2p + w_1)z_1' \geq 0$. (3) If $2p' < w_1'$ and $2p = w_1$, $z_1' = 0$ and we have $(2p' - w_1')(-z_1) \geq 0$. (4) If $2p' < w_1'$ and $2p < w_1$, then $z_1' = z_1 = 0$ and the term is zero.

14.4.2 Suppose production is described a linear activity model with basic activities $\mathbf{a}^1 = (3, 2, 1, -1)^\mathbf{T}$, $\mathbf{a}^2 = (2, 2, 1, -1)^\mathbf{T}$, $\mathbf{a}^3 = (4, 0, -1, -1)^\mathbf{T}$ and $(0, 4, -1, -1)^\mathbf{T}$. Find all efficient net output vectors.

Answer: The first thing to note is that $\mathbf{a}^1 > \mathbf{a}^2$, so use of $\mathbf{a}^2$ is never efficient. None of the other activity vectors dominate one another. We must consider non-negative linear combinations of the form $z_1\mathbf{a}^1 + z_3\mathbf{a}^3 + z_4\mathbf{a}^4 = (3z_1 + 4z_3, 2z_1 + 4z_4, z_1 - z_3 - z_4, -z_1 - z_3 - z_4)$. In fact, increasing any one of the z_i will decrease some component of the vector, so all choices with $z_2 = 0$ and $z_i \geq 0$ for $i \neq 2$ are efficient.

15.2.1 Consider the production function from Example 15.2.1: $f(z) = 1 + z - 1/(1 + z)$ with associated production set $Y = \{(q, -z) : q \leq f(z), z \geq 0\}$. Suppose the price vector is $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$. For which values of $\mathbf{p}$ can profit be maximized? For which values is $\pi(\mathbf{p}) = +\infty$?

Answer: Profit is $h(z) = p_1\big(1 + z - 1/(1 + z)\big) - p_2 z$. The derivative is $h'(z) = p_1\big(1 + 1/(1 + z)^2\big) - p_2$.

If $1 < p_2/p_1 < 2$, we can solve $h'(z) = 0$ and profit is maximized when $z = -1 + \sqrt{p_1/(p_2 - p_1)}$.

The case $p_2/p_1 = 1$ was covered in Example 15.2.1. Here $\pi(\mathbf{p}) = p_1 = p_2$, but profit cannot be maximized.

If $p_2/p_1 < 1$, the derivative is bounded above zero. It follows that profit becomes arbitrarily large as $z \to \infty$, so $\pi(\mathbf{p}) = +\infty$.

Finally, if $p_2/p_1 \geq 2$, $h'/p_1 = 1 + 1/(1 + z)^2 - p_2/p_1 \leq -1 + 1/(1 + z)^2 < 0$ for $z > 0$. In this case, profit decreases as z increases and the maximum profit is at $z = 0$. Putting the cases together, we find profit can be maximized if and only if $p_2/p_1 > 1$ while the profit function is finite for $p_2/p_1 \geq 1$.