Homework #5

15.3.2 Let $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ be defined by $f(\mathbf{z}) = \mathbf{a}\cdot\mathbf{z}$ where $\mathbf{a} \gg \mathbf{0}$. Let F be the corresponding homogenized production function. Determine the formula for F.

Answer: When $z_{L+1} > 0$ we have $F(\mathbf{z}, z_{L+1}) = z_{L+1}\mathbf{a}\cdot(\mathbf{z}/z_{L+1}) = \mathbf{a}\cdot\mathbf{z} = f(\mathbf{z})$. We take the limit as $z_{L+1} \to 0$ to find $F(\mathbf{z}, 0)$, so $F(\mathbf{z}, 0) = \mathbf{a}\cdot\mathbf{z} = f(\mathbf{z})$.

15.3.3 Let $f\colon \mathbb{R}^L_+ \to \mathbb{R}_+$ be continuous, concave, and constant returns to scale. Let F be the corresponding homogenized production function. Show that $F(\mathbf{z}, 0) = f(\mathbf{z})$.

Answer: When $z_{L+1} > 0$ we have $F(\mathbf{z}, z_{L+1}) = z_{L+1}f(\mathbf{z}/z_{L+1}) = z_{L+1}/z_{L+1}f(\mathbf{z}) = f(\mathbf{z})$. We take the limit as $z_{L+1} \to 0$ to find $F(\mathbf{z}, 0)$, so $F(\mathbf{z}, 0) = f(\mathbf{z}) = f(\mathbf{z})$.

15.3.6 Suppose a diminishing returns production set Y is also a constant returns production set. Find the augmented production set $\hat{Y}$. What price do you expect the entrepreneurial factor to have? Why?

Answer: The augmented production set is given by $\hat{Y} = \text{cl}\{(z\mathbf{y}, -z) : \mathbf{y} \in Y,\ z > 0\}$. Note that if $\mathbf{y} \in zY$ for any $z > 0$, then $\mathbf{y} \in z'Y$ for all $z' > 0$. Then $\hat{Y} = \overline{(Y \times (-\mathbb{R}_{++})} = Y \times (-\mathbb{R}_+)$.

Since the entrepreneurial factor is not needed to allow a firm using $\hat{Y}$ to operate at any scale, it will not be demanded at any positive price, and we expect the price of the entrepreneurial factor to be zero.

16.3.1 Consider a two-person, two-good exchange economy. Endowments are $\boldsymbol{\omega}^1 = (1, 2)$ and $\boldsymbol{\omega}^2 = (1, 3)$ and utility is $u_1(\mathbf{x}^1) = x^1_1 + 2x^1_2$ and $u_2(\mathbf{x}^2) = \sqrt{x^2_1 x^2_2}$. Find all Walrasian equilibrium prices and allocations.

Answer: Since consumer two has Cobb-Douglas utility, both prices will have to be positive in equilibrium. We normalize so $(p_1, p_2) = (1, p)$. Then consumer incomes are $m^1 = 1 + 2p$ and $m^2 = 1 + 3p$. Consumer two has equally weighted Cobb-Douglas utility, so $\mathbf{x}^2(p) = (1 + 3p)(1/2, 1/2p)$. Consumer one has linear utility, and will be at a corner unless $p = 2$. Consumer one's demand is:

$$\mathbf{x}^1(p) = \begin{cases} (1 + 2p, 0) & \text{if } p > 2 \\ \{(x, (5 - x)/2) : 0 \le x \le 5\} & \text{if } p = 2 \\ (0, (1 + 2p)/p) & \text{if } p < 2. \end{cases}$$

where we have used the fact that $m^1 = 5$ when $p = 2$.

The aggregate endowment is $\boldsymbol{\omega} = (2, 5)$. If $p > 2$, only consumer two demands good 2 and market clearing requires $(1 + 3p)/2p = 5$. Then $p = 1/7$, contradicting the fact that $p > 2$. If $p < 2$, only consumer one demands good 1 and market clearing requires $(1+3p)/2 = 2$. Then $p = 1$ is an equilibrium. The corresponding allocation is $\mathbf{x}^1 = (0, 3)$ and $\mathbf{x}^2 = (2, 2)$. Finally, if $p = 2$, $\mathbf{x}^2 = (7/2, 7/4)$, meaning there is excess demand for good 1. This is not an equilibrium.