Homework #6

16.3.4 Consider a three-person, three-person exchange economy. Endowments are $\boldsymbol{\omega}^1 = (1,1,0)$, $\boldsymbol{\omega}^2 = (0,1,0)$ and $\boldsymbol{\omega}^3 = (0,1,1)$. Utility is $u_1(\mathbf{x}^1) = (x_1^1)^{1/2}(x_2^1)^{1/3}(x_3^1)^{1/6}$, $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{1/3}(x_3^2)^{1/3}$, and $u_3(\mathbf{x}^3) = (x_1^3)^{1/3}(x_2^3)^{1/2}(x_3^3)^{1/6}$. Find all Walrasian equilibrium prices and allocations.

Answer: Each consumer is Cobb-Douglas, so no price will be zero in equilibrium. We can safely pick good 1 as numéraire. Income is then $m^1 = 1 + p_2$, $m^2 = p_2$, and $m^3 = p_2 + p_3$. The individual demand functions are $\mathbf{x}^1(\mathbf{p}) = (m^1/6)(3, 2/p_2, 1/p_3)$, $\mathbf{x}^2(\mathbf{p}) = (m^2/6)(2, 2/p_2, 2/p_3)$, and $\mathbf{x}^3(\mathbf{p}) = (m^3/6)(2, 3/p_2, 1/p_3)$. Aggregating and setting supply equal to demand yields

$$\begin{aligned} 1 &= x_1(\mathbf{p}) = (3 + 7p_2 + 2p_3)/6 \\ 3 &= x_2(\mathbf{p}) = (2 + 7p_2 + 3p_3)/6 \\ 1 &= x_3(\mathbf{p}) = (1 + 4p_2 + p_3)/6. \end{aligned}$$

These can be simplified, yielding

$$\begin{aligned} 3 &= 7p_2 + 2p_3 \\ 2 &= 11p_2 - 3p_3 \\ 1 &= -4p_2 + 5p_3. \end{aligned}$$

One of the equations is redundant, as can be see by adding the last two to get the first one. Solving the first two equations gives us the normalized equilibrium price vector $(1, 13/43, 19/43)$. Then $m^1 = 56/43$, $m^2 = 13/43$, and $m^3 = 32/43$. This results in the following allocations of goods:

$$\begin{aligned} \mathbf{x}^1 &= \left(\frac{84}{129}, \frac{56}{39}, \frac{28}{57}\right) = \left(\frac{28}{43}, \frac{56}{39}, \frac{28}{57}\right) \\ \mathbf{x}^2 &= \left(\frac{13}{129}, \frac{13}{39}, \frac{13}{57}\right) = \left(\frac{13}{129}, \frac{1}{3}, \frac{13}{57}\right) \\ \mathbf{x}^3 &= \left(\frac{32}{129}, \frac{48}{39}, \frac{16}{57}\right) = \left(\frac{32}{129}, \frac{16}{13}, \frac{16}{57}\right) \end{aligned}$$

16.4.1 Consider a two-agent, two-good, one-firm production economy where utility is $u_1(\mathbf{x}^1) = (x_1^1)^{1/2}(x_2^1)^{1/2}$ and $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$, and endowments are $\boldsymbol{\omega}^1 = (3, 0)$ and $\boldsymbol{\omega}^2 = (6, 0)$. There is one firm with production set $Y = \{\mathbf{y} : y_1 \leq 0, y_2 \leq -y_1\}$. Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

Answer: If the firm does not produce anything, the economy will not have any of good two. Since demand for good two will be positive, this cannot be an equilibrium.

If the firm does produce ($y_2 > 0$), constant returns to scale implies that profits will be zero. This requires $p_1 = p_2$. We can now normalize prices so $\mathbf{p} = (1, 1)$, which are the equilibrium prices. Incomes are then $m^1 = 3$ and $m^2 = 6$. The Cobb-Douglas utilities yield demands $\mathbf{x}^1 = (3/2, 3/2)$ and $\mathbf{x}^2 = (2, 4)$. All that is left is to find the production vector. Now $\mathbf{x}^1 + \mathbf{x}^2 = \boldsymbol{\omega} + \mathbf{y}$ by market clearing. This can be rewritten $(7/2, 11/2) = (9, 0) + (-y_2, y_2)$. It follows that the production vector is $\mathbf{y} = (-11/2, +11/2)$.

16.4.4 Consider a two-agent, two-good, one-firm production economy where utility is $u_1(\mathbf{x}^1) = (x_1^1)^{1/2}(x_2^1)^{1/2}$ and $u_2(\mathbf{x}^2) = (x_1^2)^{1/2}(x_2^2)^{1/2}$, and endowments are $\boldsymbol{\omega}^1 = (4, 0)$ and $\boldsymbol{\omega}^2 = (4, 0)$. Each agent will receive half of the profits of the firm. There is one firm with production set $Y = \{\mathbf{y} : y_1 \leq 0, y_2 \leq \sqrt{-y_1}\}$. Find the equilibrium prices, equilibrium demands by individuals, and the firm's equilibrium net output.

Answer: We start by considering profit, $p_2\sqrt{-y_1} + p_1 y_1$. The first-order conditions for profit maximization are $(p_2/2)(1/\sqrt{-y_1}) = p_1$, so $-p_2^2/4p_1^2 = y_1$, $p_2/2p_1 = y_2$, and $\pi(\mathbf{p}) = p_2^2/4p_1$. Each consumer receives half of the profit and has endowment income $4p_1$, so each consumer has income $m = 4p_1 + p_2^2/8p_1$. Since consumers have identical preferences, and income, they will consume the same amount. Utility is equal-weighted Cobb-Douglas, and market demand is $2m(1/2p_1, 1/2p_2) = m(1/p_1, 1/p_2)$. Adding the endowment to the firm's supply yields $(8 - p_2^2/4p_1^2, p_2/2p_1)$. Both goods are demanded in equilibrium, and prices must be strictly positive. We normalize prices so that $p_1 = 1$ and $p_2 = p$. Then market clearing for good two says $p/2 = m/p = (4 + p^2/8)/p$. Thus $4p^2 = 32 + p^2$ implying $p = \pm\sqrt{32/3}$. Only the positive root makes economic sense, and the equilibrium has $\mathbf{p} = (1, 4\sqrt{2/3})$, $\mathbf{y} = (-8/3, \sqrt{8/3})$, $m = 16/3$, and $\mathbf{x}^1 = \mathbf{x}^2 = (8/3, \sqrt{2/3})$.

17.2.3 Suppose an exchange economy has 2 consumers and 2 goods. Consumer one has endowment $\boldsymbol{\omega}^1 = (1, 0)$. Utility is $u_1(\mathbf{x}^1) = \sqrt{x_1^1}$. Consumer two has endowment $\boldsymbol{\omega}^2 = (0, 1)$. Utility is $u_2(\mathbf{x}^2) = \sqrt{x_1^2} + \sqrt{x_2^2}$. The consumption sets are $\mathfrak{X}_i = \mathbb{R}_+^2$.

a) Show that there is no competitive equilibrium.

b) Which hypotheses of the Equilibrium Existence Theorem are violated. Make sure you list all of them.

Answer:

a) We first calculate demands. Since consumer two's utility is strictly increasing in each good, prices must be strictly positive in equilibrium.

Since consumer one only values good one, $\mathbf{x}^1(\mathbf{p}) = (1, 0)$.

Consumer two has income p_2. Setting the marginal rate of subsitution equal to the relative price, we find $p_1/p_2 = \sqrt{x_2^2}/\sqrt{x_1^2}$. It follows that

$$p_1 x_1^2 + p_2 x_2^2 = \frac{p_1 + p_2}{p_2} p_1 x_1^2.$$

Setting this equal to income, we find $x_1^2 = p_2^2/p_1(p_1 + p_2)$ and $x_2^2 = p_1/(p_1 + p_2)$.

In equilibrium, we must have $x_2^2 = 1$, implying $p_2 = 0$, which is impossible. There cannot be an equilibrium.

b) Two hypotheses are violated. (1) Consumer one's utility function is not strictly concave, although consumer two's utility function is strictly concave. (2) The endowments of both consumers are not strictly positive. Since we still have a demand function for consumer one, the former violation is unimportant. The latter is responsible for the non-existence of equilibrium.