Homework #7

19.1.1 Suppose $u_1(\mathbf{x}^1) = (x_1^1)^{1/3}(x_2^1)^{2/3}$, $u_2(\mathbf{x}^2) = (x_1^2)^{1/3}(x_2^2)^{2/3}$, and $\boldsymbol{\omega} = (10, 2)$. Find all Pareto optimal allocations.

Answer: Since both consumers have strictly monotonic preferences, all goods will be consumed. Thus we may presume that $x_1^2 = 10 - x_1^1$ and $x_2^2 = 2 - x_2^1$. Further, because both consumers have Cobb-Douglas preferences, the only corner solutions will the be obvious ones ($\mathbf{x}^i = \boldsymbol{\omega}$ and $\mathbf{x}^j = \mathbf{0}$ for $i \neq j$). This means that we can use the simplified Lagrangian $\mathcal{L} = (x_1^1)^{1/3}(x_2^1)^{2/3} + \lambda\big((10 - x_1^1)^{1/3}(2 - x_2^1)^{2/3} - \bar{u}_2\big)$.

The first-order conditions are then

$$\frac{1}{3}\left(\frac{x_2^1}{x_1^1}\right)^{2/3} = \lambda\frac{1}{3}\left(\frac{2 - x_2^1}{10 - x_1^1}\right)^{2/3}$$
$$\frac{2}{3}\left(\frac{x_1^1}{x_2^1}\right)^{1/3} = \lambda\frac{2}{3}\left(\frac{10 - x_1^1}{2 - x_2^1}\right)^{1/3}$$

Eliminating λ, we find

$$\frac{x_2^1}{x_1^1} = \frac{2 - x_2^1}{10 - x_1^1},$$

which implies $5x_2^1 = x_1^1$.

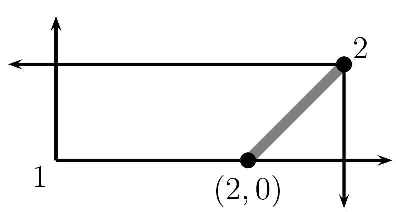
The Pareto set is

$$\left\{\alpha\begin{pmatrix}10\\2\end{pmatrix}, (1-\alpha)\begin{pmatrix}10\\2\end{pmatrix} : 0 \le \alpha \le 1\right\}.$$

The corresponding utility levels are $u_1 = 2 \cdot 5^{1/3}\alpha$ and $u_2 = 2 \cdot 5^{1/3}(1 - \alpha)$.

19.1.6 Suppose $u_1(\mathbf{x}^1) = x_1^1 + 2x_2^1$ and $u_2(\mathbf{x}^2) = \min\{x_1^2, x_2^2\}$, with $\boldsymbol{\omega} = (3, 1)$. Find all Pareto optimal allocations.

Answer: $\text{MRS}_{12}^1 = 1/2$ while MRS_{12}^2 can be interpreted as anything when $x_1^2 = x_2^2$. The interior Pareto optimal allocations run from the upper right corner of the box to $(2, 0)$. The two boundary points are included. Note that $(x, 0)$ for $x < 2$ is not Pareto optimal as $(2, 0)$ is a Pareto improvement (consumer 1 is better off, consumer 2 is indifferent). The set of Pareto optima is the heavy line in the diagram, $\{(x_1, x_2) : x_1 - 2 = x_2, x_1 \ge 2\}$.



19.2.1 An economy has two goods and two identical Cobb-Douglas consumers with $u_i(\mathbf{x}^i) = \sqrt{x_1^i x_2^i}$. The total endowment is $(0, 6)$. There is one constant returns to scale firm that produces good 1 and uses good 2 as its only input. The production function is $f(z) = 2z$.

Find all Pareto optimal allocations of goods and the corresponding net output vector.

Answer: The net output vector will have the form $\mathbf{y} = (-2y_2, y_2)$ with $y_2 \leq 0$. Here the marginal rate of transformation is $\text{MRT}_{12} = 1/2$. This must also be the marginal rate of substitution at any interior Pareto optimum. Note that since utility is zero if there is no production, the production technology must be used.

Now $\text{MRS}_{12}^i = x_2^i/x_1^i = 1/2$, so $2x_2^i = x_1^i$. Summing over both consumers, aggregate consumption obeys $2x_2 = x_1$. Since good 1 can only be obtained from the production sector, $x_1 = -2y_2$ and $x_2 = 6 + y_2$. Thus $x_2 = -y_2$ and $x_2 = 6 + y_2$. It follows that $x_2 = 3$ and $y_2 = -3$, so $x_1 = 6$.

Since both consumers will consume in the same proportions, $\mathbf{x}_1 = \alpha(6, 3)$ and $\mathbf{x}_2 = (1-\alpha)(6, 3)$ for some α between 0 and 1. Also, $\mathbf{y} = (6, -3)$. These are the Pareto optimal allocations.

20.1.3 An economy has two goods and two identical Cobb-Douglas consumers with $u_i(\mathbf{x}^i) = \sqrt{x_1^i x_2^i}$. The total endowment is $(0, 6)$. There is one constant returns to scale firm that produces good 1 and uses good 2 as its only input. The production function is $f(z) = 6z^{1/2}$.

a) Find all Pareto optimal allocations.

b) Find all envy-free Pareto optimal allocations.

c) If possible, find a price vector and wealth levels that yield an envy-free allocation in equilibrium.

d) Suppose that each consumer gets one-half of the profits. Which endowments $(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ will yield an envy-free allocation as an equilibrium?

Answer:

a) We will need to use the technology to produce good one, otherwise utility of both consumers will be zero. The marginal rate of transformation is $\text{MRT}_{12} = \sqrt{z}/3$. To see this, note that $y_1 \leq 6\sqrt{-y_2}$. Squaring, we obtain $y_1^2 \leq -36y_2$, so the transformation function is $T(y_1, y_2) = y_1^2 + 36y_2 \leq 0$. Then marginal rate of transformation is $T_{12} = 2y_1/36 = y_1/18 = 6\sqrt{z}/18 = \sqrt{z}/3$.

For consumers, $\text{MRS}_{12}^i = x_2^i/x_1^i$. At any interior Pareto optimum, both consumers will have the same MRS_{12}, in which case $\text{MRS}_{12} = x_2/x_1$. Feasibility requires $x_2 =$

$6 - z$ and $x_1 = 6\sqrt{z}$. Thus

$$\frac{\sqrt{z}}{3} = \mathrm{MRT}_{12} = \mathrm{MRS}_{12} = \frac{6-z}{6\sqrt{z}}$$

so $2z = 6 - z$. It follows that $z = 2$. Overall net output is $\mathbf{y} = (6\sqrt{2}, -2)$ which leaves $\boldsymbol{\omega} + \mathbf{y} = (6\sqrt{2}, 4)$ available for consumption.

Both consumers consume goods in the same proportions, meaning that each consumer consumes some fraction of the overall endowment. We then have $\mathbf{x}^1 = \alpha(6\sqrt{2}, 4)$ and $\mathbf{x}^2 = (1-\alpha)(6\sqrt{2}, 4)$ for some α, $0 \le \alpha \le 1$.

b) Using the Pareto optimum above, $u_1(\mathbf{x}^1) = 2\alpha 3^{1/2}2^{3/4}$ and $u_2(\mathbf{x}^2) = 2(1-\alpha)3^{1/2}2^{3/4}$. For the allocation to be envy-free, it must obey $u_1(\mathbf{x}^1) \ge u_1(\mathbf{x}^2)$ and $u_2(\mathbf{x}^2) \ge u_2(\mathbf{x}^1)$. Since both consumers have the same utility function, this requires $\alpha = 1/2$. The only envy-free allocation is $\mathbf{x}^1 = \mathbf{x}^2 = (3\sqrt{2}, 2)$.

c) We already know the price vector must have $p_1/p_2 = \mathrm{MRT}_{12} = \sqrt{2}/3$. The vector $\mathbf{p} = (\sqrt{2}, 3)$ (or any positive multiple) will do. Then $\mathbf{p}\cdot\mathbf{x}^1 = \mathbf{p}\cdot\mathbf{x}^2 = 12$, so $m^1 = m^2 = 12$ gives the wealth levels that yield the envy-free allocation.

d) The endowments must be non-negative and obey $\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2 = (0, 6)$, so $\omega_1^1 + \omega_1^2 = 0$ and $\omega_2^1 + \omega_2^2 = 6$. Clearly $\omega_1^1 = \omega_1^2 = 0$. The wealth levels must be the same, so $\omega_1^1 = \omega_2^2 = 3$. Then endowment income is \$9 for each consumer. Profit is $(\sqrt{2}, 3)\cdot(6\sqrt{2}, -2) = 6$, so each consumer gets \$3 of profit income, for a total of \$12.