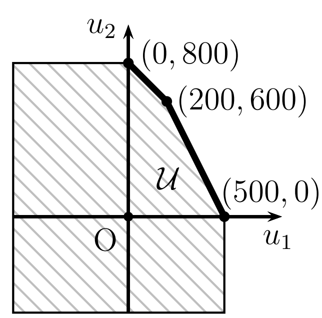
Homework #8

20.2.1 Suppose there are two goods and two consumers, both with consumption set $\mathfrak{X} = \mathbb{R}^2_+$. The aggregate endowment is $\boldsymbol{\omega} = (200, 300)$. Consumer one has utility function $u_1(x_1, x_2) = x_1 + x_2$ and consumer two's utility is $u_2(x_1, x_2) = x_1 + 2x_2$. Find the utility possibility set.

Answer: We start by computing the Pareto frontier. Notice that $\text{MRS}^1_{12} = 1$ while $\text{MRS}^2_{12} = 1/2$. There are no interior Pareto optima. Consumer one puts a higher relative value on good 1, while consumer 2 puts a higher relative value on good 2. Good 1 should be preferentially allocated to consumer one, and good 2 to consumer two. Starting at $\mathbf{x}^1 = (0, 0)$ and $\mathbf{x}^2 = (200, 300)$, where $u_1 = 0$ and $u_2 = 800$, we begin by shifting good 1 to consumer one until we get to $\mathbf{x}^1 = (200, 0)$ and $\mathbf{x}^2 = (0, 300)$ where $u_1 = 200$ and $u_2 = 600$. At this point we start shifting good 2 to consumer 1 until we get to the other corner of the Edgeworth box, where $u_1 = 500$ and $u_2 = 0$.

Thus $\mathcal{U}$ is the comprehensive hull of the points $(0, 800)$, $(200, 600)$, and $(500, 0)$.



Here $\mathcal{U}$ is the utility possibility set for Problem 20.2.1. The heavy line indicates the Pareto frontier.

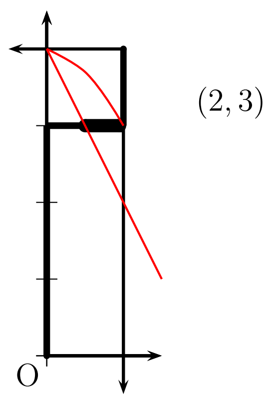
21.1.2 An exchange economy has two consumers with utility $u_1(\mathbf{x}^1) = (x^1_1 x^1_2 x^1_3)^{1/3}$ and $u_2(\mathbf{x}^2) = (x^2_1 x^2_2 x^2_3)^{1/3}$. Their endowments are $\boldsymbol{\omega}^1 = (2, 1, 1)$ and $\boldsymbol{\omega}^2 = (3, 1, 2)$. Find the core.

Answer: Here we have identical Cobb-Douglas preferences. The aggregate endowment is $\boldsymbol{\omega} = (5, 2, 3)$. Pareto optimal allocations will have the form $\mathbf{x}^1 = \alpha\boldsymbol{\omega}$ and $\mathbf{x}^2 = (1-\alpha)\boldsymbol{\omega}$ where $0 \le \alpha \le 1$.

We must also satisfy the requirement of individual rationality. This requires $u_1(\mathbf{x}^1) \ge u_1(2, 1, 1) = 2^{1/3}$ and $u_2(\mathbf{x}^2) \ge u_2(3, 1, 2) = 6^{1/3}$. Since $u_1(\alpha\boldsymbol{\omega}) = \alpha 30^{1/3}$ and $u_2\big((1-\alpha)\boldsymbol{\omega}\big) = (1-\alpha)30^{1/3}$, the core allocations are the Pareto optima that obey $(1/15)^{-1/3} \le \alpha \le 1 - (1/5)^{-1/3}$. The range of α is fairly narrow, about 0.4055–0.4152.

21.1.8 An exchange economy has two consumers with utility functions $u_1(\mathbf{x}^1) = 2x_1^1 + x_2^1$ and $u_2(\mathbf{x}^2) = x_1^2 + \sqrt{x_2^2}$. Their endowments are $\boldsymbol{\omega}^1 = (0, 4)$ and $\boldsymbol{\omega}^2 = (1, 0)$. Find the core.

Answer: We first find the interior Pareto optima. Here $\text{MRS}_{12}^1 = 2$ and $\text{MRS}_{12}^2 = 2\sqrt{x_2^2}$. Thus all interior Pareto optima obey $x_2^2 = 1$ and $x_2^1 = 3$. For non-interior optima, consideration of the marginal rates of substitution shows there are two vertical segments of interst. There is the vertical segment from $\mathbf{x}^1 = (0, 0)$ to $(0, 3)$, where consumer two gets all of good one, and the vertical segment from $\mathbf{x}^1 = (1, 3)$ to $(1, 4)$, where consumer one gets all of good one. The Pareto optima are shown by the medium heavy line in the diagram.



Individual rationality requires $u_1(\mathbf{x}^1) \geq 4$. This crosses the $x_2^1 = 3$ line at $\mathbf{x}^1 = (1/2, 3)$. Individual rationality also requires $u_2(\mathbf{x}^2) \geq 1$, which crosses the x_2^2 line at $\mathbf{x}^2 = (0, 1)$ (or $\mathbf{x}^1 = (1, 3)$). It follows that the core is $\{(\mathbf{x}^1, \mathbf{x}^2) : x_2^1 = 3, x_1^2 = 1,\ 1/2 \leq x_1^1 \leq 1$, and $x_1^2 = 1 - x_1^1\}$. The two red lines illustrate the individual rationality constraints, and the portion of the Pareto set in-between them is the core, marked with a very heavy line.

21.1.11 An exchange economy has three consumers with utility functions $u_i(\mathbf{x}) = \sqrt{x_1^i x_2^i}$ for $i = 1, 2, 3$. Their endowments are $\boldsymbol{\omega}^1 = (1, 2)$, $\boldsymbol{\omega}^2 = (1, 3)$, and $\boldsymbol{\omega}^3 = (4, 1)$. Find the core.

Answer: Again we have identical Cobb-Douglas utility, and the Pareto set (in utility space) is $\{(u_1, u_2, u_3) \in \mathbb{R}_+^3 : u_1 + u_2 + u_3 = 6\}$. We additionally have to satisfy individual rationality: $u_1 \geq \sqrt{2}$, $u_2 \geq \sqrt{3}$ and $u_3 \geq 2$. We also must be at least as well off as in the Pareto optima for 2-consumer coalitions. Thus $u_1 + u_2 \geq \sqrt{10}$, $u_1 + u_3 \geq \sqrt{15}$ and $u_2 + u_3 \geq \sqrt{20}$.

We can simplify the conditions by substituting $u_3 = 6 - u_1 - u_2$. Then we obtain: $\sqrt{2} \leq u_1 \leq 6 - \sqrt{20}$, $\sqrt{3} \leq u_2 \leq 6 - \sqrt{15}$, and $\sqrt{10} \leq u_1 + u_2 \leq 4$ together with

$u_3 = 6 - u_1 - u_2$.

Any $u_i \geq 0$ that meet the above conditions, such as $(1\frac{1}{2}, 2, 2\frac{1}{2})$ are core utility allocations. The corresponding goods allocations are $\mathbf{x}^i = u_i(1, 1)$.